

On the \mathcal{R} -boundedness for the two phase problem with phase transition: compressible-incompressible model problem

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Abstract

In this paper, we prove the maximal L_p - L_q regularity of the compressible and incompressible two phase flow with phase transition in the model problem case with the help of \mathcal{R} -bounded solution operators corresponding to generalized resolvent problem. The problem arises from the mathematical study of the motion of two-phase flows having gaseous phase and liquid phase separated by a sharp interface with phase transition. Using the result obtained in this paper, in [10] we proved the local well-posedness of free boundary problem for the compressible and incompressible two phase flow separated by sharp interface with phase transition.

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1 Introduction

In this paper, we prove the maximal L_p - L_q regularity of the compressible and incompressible two phase flow with phase transition in the model problem case with the help of \mathcal{R} -bounded solution operators corresponding to generalized resolvent problem. The problem arises from the mathematical study of the motion of two-phase flows having gaseous phase and liquid phase separated by a sharp interface with phase transition, which is formulated as follows: Let \mathbb{R}^N be the N dimensional Euclidean space. Let Ω_- be a domain in \mathbb{R}^N with boundary Γ , which is occupied by the liquid. Set $\Omega_+ = \mathbb{R}^N - \overline{\Omega_-}$, where $\overline{\Omega_-}$ stands for the closure of domain Ω_- , which is occupied by the gas. Let $\varphi = \varphi(\xi, t) = (\varphi_1(\xi, t), \dots, \varphi_N(\xi, t))$ be a function defined on \mathbb{R}^N for each $t \in (0, T)$, $\xi = (\xi_1, \dots, \xi_N)$ being the reference coordinate system. We assume that the correspondence: $\xi \rightarrow \varphi(\xi, t)$ is one to one map from \mathbb{R}^N onto itself for each $t \in (0, T)$. Set $(\partial_t \varphi)(\xi, t) = \mathbf{v}(x, t)$ with $x = \varphi(\xi, t)$,

$$\Omega_{\pm}(t) = \{x = \varphi(\xi, t) \mid \xi \in \Omega_{\pm}\}, \quad \Gamma(t) = \{x = \varphi(\xi, t) \mid \xi \in \Gamma\}.$$

Let $\mathbf{n}_{\Gamma(t)}$ be the unit outer normal to $\Gamma(t)$ pointed from $\Omega_-(t)$ to $\Omega_+(t)$. For any $x_0 \in \Gamma(t)$, we set

$$[[v]](x_0) = \lim_{\substack{x \rightarrow x_0 \\ x \in \Omega_-(t)}} v(x) - \lim_{\substack{x \rightarrow x_0 \\ x \in \Omega_+(t)}} v(x) \quad (\text{the jump of } v \text{ across } \Gamma(t))$$

for any v defined on $\dot{\Omega}(t) = \Omega_+(t) \cup \Omega_-(t)$. In the sequel, we write $v_{\pm} = v|_{\Omega_{\pm}(t)}$. Moreover, given v_{\pm} defined on $\Omega_{\pm}(t)$, we define v by $v(x) = v_{\pm}(x)$ for $x \in \Omega_{\pm}(t)$.

Let $\mathbf{u} : \dot{\Omega}(t) \rightarrow \mathbb{R}^N$ be the velocity fields, $\rho : \dot{\Omega}(t) \rightarrow (0, \infty)$ the mass field, $\pi : \dot{\Omega}(t) \rightarrow \mathbb{R}$ the pressure field, $\mathbf{T} : \dot{\Omega}(t) \rightarrow \mathbb{R}^N$ the stress tensor field, $\theta : \dot{\Omega}(t) \rightarrow (0, \infty)$ the thermal fields, $\psi : \dot{\Omega}(t) \rightarrow \mathbb{R}$ the

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free energy, $\eta : \dot{\Omega}(t) \rightarrow \mathbb{R}$ the entropy, H_Γ the mean curvature of $\Gamma(t)$, κ the specific heat, d the thermal conductivity and j the phase flux.

We assume that $\rho_+ \neq \rho_-$. Then, the motion of two-phase flows having gaseous phase and liquid phase separated by a sharp interface with phase transition is described as follows:

$$\begin{cases} \rho_+(\partial_t \mathbf{u}_+ + \mathbf{u}_+ \cdot \nabla \mathbf{u}_+) - \text{Div } \mathbf{T}_+ = 0, & \partial_t \rho_+ + \text{div}(\rho_+ \mathbf{u}_+) = 0, \\ \rho_+ \kappa_+(\partial_t \theta_+ + \mathbf{u}_+ \cdot \nabla \theta_+) - \text{div}(d_+ \nabla \theta_+) - \mathbf{T}_+ : \nabla \mathbf{u}_+ - \frac{\pi_+}{\rho} \text{div } \mathbf{u}_+ = 0 & \text{for } x \in \Omega_+(t), t > 0, \end{cases} \quad (1.1)$$

$$\begin{cases} \rho_{*-}(\partial_t \mathbf{u}_- + \mathbf{u}_- \cdot \nabla \mathbf{u}_-) - \text{Div } \mathbf{T}_- = 0, & \text{div } \mathbf{u}_- = 0, \\ \rho_{*-} \kappa_-(\partial_t \theta_- + \mathbf{u}_- \cdot \nabla \theta_-) - \text{div}(d_- \nabla \theta_-) - \mathbf{T}_- : \nabla \mathbf{u}_- = 0 & \text{for } x \in \Omega_-(t), t > 0 \end{cases}$$

subject to the interface conditions: for $x \in \Gamma(t)$ and $t > 0$,

$$\begin{cases} [[\frac{1}{\rho}]] j^2 \mathbf{n}_\Gamma - [[\mathbf{T} \mathbf{n}_{\Gamma(t)}]] = -\sigma H_\Gamma \mathbf{n}_{\Gamma(t)}, & [[\mathbf{u} - (\mathbf{u} \cdot \mathbf{n}_{\Gamma(t)}) \mathbf{n}_{\Gamma(t)}]] = 0, \\ j[[\theta \eta]] - [[d(\nabla \theta) \cdot \mathbf{n}_{\Gamma(t)}]] = 0, & [[\theta]] = 0, \\ [[[\psi]] + [[\frac{1}{2\rho^2}]] j^2 - [[\frac{1}{\rho} \mathbf{n}_{\Gamma(t)} \cdot \mathbf{T} \mathbf{n}_{\Gamma(t)}]]] = 0, & \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} = \frac{[[\rho \mathbf{u}]] \cdot \mathbf{n}_{\Gamma(t)}}{[[\rho]]}, \\ j = \frac{[[\rho \mathbf{u}]] \cdot \mathbf{n}_{\Gamma(t)}}{[[\rho]]}, \end{cases} \quad (1.2)$$

and the initial conditions:

$$\begin{aligned} (\rho_+, \mathbf{u}_+, \theta_+)|_{t=0} &= (\rho_{*+} + \rho_{0+}, \mathbf{u}_{0+}, \theta_* + \theta_{0+}) \text{ in } \Omega_+, \\ (\mathbf{u}_-, \theta_-)|_{t=0} &= (\mathbf{u}_{0+}, \theta_* + \theta_{0+}) \text{ in } \Omega_-, \quad h|_{t=0} = h_0 \text{ on } \Gamma. \end{aligned} \quad (1.3)$$

Here, $\rho_{*\pm}$, θ_* and σ are positive constants describing the reference mass densities of Ω_\pm , the reference temperature of both Ω_\pm and the coefficient of the surface tension, respectively. Moreover, $\mathbf{T}_\pm = \mathbf{S}_\pm - \pi_\pm \mathbf{I}$ with

$$\begin{aligned} \mathbf{S}_+ &= \mathbf{S}_+(\mathbf{u}_+, \rho_+, \theta_+) = \mu_+ \mathbf{D}(\mathbf{u}_+) + (\nu_+ - \mu_+) \text{div } \mathbf{u}_+ \mathbf{I}, \\ \mathbf{S}_- &= \mathbf{S}_-(\mathbf{u}_-, \theta_-) = \mu_- \mathbf{D}(\mathbf{u}_-); \end{aligned}$$

for any scalar field θ we set $\nabla \theta = (\partial_1 \theta, \dots, \partial_N \theta)$, where $\partial_j = \partial/\partial x_j$; for any vector field $\mathbf{u} = (u_1, \dots, u_N)$ $\nabla \mathbf{u}$ is the $N \times N$ matrix whose (i, j) component is $\partial_i u_j$, $\mathbf{D}(\mathbf{u})$ the deformation tensor whose (j, k) components are $D_{jk}(\mathbf{u}) = \frac{1}{2}(\partial_j u_k + \partial_k u_j)$ and $\text{div } \mathbf{u} = \sum_{j=1}^N \partial_j u_j$; and \mathbf{I} is the $N \times N$ identity matrix. Finally, for any matrix field \mathbf{K} with components K_{ij} , the quantity $\text{Div } \mathbf{K}$ is an N -vector with i -component $\sum_{j=1}^N \partial_j K_{ij}$ and $\mathbf{K} : \nabla \mathbf{u} = \sum_{i,j=1}^N K_{ij} \partial_i u_j$.

We assume that d , μ , ν_+ , κ , ψ and η are given as follows: $d_+ = d_+(\rho, \theta)$, $\mu_+ = \mu_+(\rho, \theta)$, $\nu_+ = \nu_+(\rho, \theta)$, $\kappa_+ = \kappa_+(\rho, \theta)$ are positive C^∞ functions with respect to $(\rho, \theta) \in (0, \infty) \times (0, \infty)$, and $\psi_+ = \psi_+(\theta, \rho)$ and $\eta_+ = \eta_+(\theta, \rho)$ are real valued C^∞ functions with respect to $(\rho, \theta) \in (0, \infty) \times (0, \infty)$, while $d_- = d_-(\theta)$, $\mu_- = \mu_-(\theta)$, $\kappa_- = \kappa_-(\theta)$ are positive C^∞ functions with respect to $\theta \in (0, \infty)$, and $\psi_- = \psi_-(\theta)$ and $\eta_- = \eta_-(\theta)$ are real valued C^∞ functions with respect to $\theta \in (0, \infty)$. And also, we assume that π_+ is given by $\pi_+ = P_+(\rho, \theta)$, where P_+ is a C^∞ function with respect to $(\rho, \theta) \in (0, \infty) \times (0, \infty)$ such that $\frac{\partial P_+}{\partial \rho} > 0$ for any $(\rho, \theta) \in (0, \infty) \times (0, \infty)$.

Since we prove the local well-posedness of problem (1.1), (1.2) and (1.3) with the help of the maximal L_p - L_q regularity results for the linearized equations, representing ρ_+ by the integration of the equation: $\partial_t \rho_+ + \text{div}(\rho_+ \mathbf{u}_+) = 0$ along the characteristic curve generated by \mathbf{u}_+ we eliminate ρ_+ from the equations in $\Omega_+(t)$ and $\Gamma(t)$ ¹, so that we have the nonlinear parabolic equations. After this procedure, as the linearized problem we have the following two problems as model problems: In the sequel, for any $x_0 \in \mathbb{R}_0^N$ we define $f|_\pm(x_0)$ by

$$f|_\pm(x_0) = \lim_{\substack{x \rightarrow x_0 \\ x \in \mathbb{R}_\pm^N}} f(x),$$

¹The idea follows from Tani [16].

where we have set

$$\mathbb{R}_\pm^N = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N \mid \pm x_N > 0\}, \quad \mathbb{R}_0^N = \{x \in \mathbb{R}^N \mid x_N = 0\}.$$

One is the interface problem for the Stokes system:

$$\begin{aligned} \rho_{*+} \partial_t \mathbf{u}_+ - \operatorname{Div} \mathbf{S}_{*+}(\mathbf{u}_+) &= \mathbf{f}_+ & \text{in } \mathbb{R}_+^N \times (0, T) \\ \rho_{*-} \partial_t \mathbf{u}_- - \operatorname{Div} \mathbf{S}_{*-}(\mathbf{u}_-) + \nabla \pi_- &= \mathbf{f}_-, \quad \operatorname{div} \mathbf{u}_- = f_{\operatorname{div}} = \operatorname{div} \mathbf{f}_{\operatorname{div}} & \text{in } \mathbb{R}_-^N \times (0, T) \end{aligned} \quad (1.4)$$

subject to the interface condition: for $x \in \mathbb{R}_0^N$ and $t \in (0, T)$

$$\begin{aligned} \mu_{*-} D_{iN}(\mathbf{u}_-)|_- - \mu_{*+} D_{iN}(\mathbf{u}_+)|_+ &= g_i \quad (i = 1, \dots, N-1), \\ (\mu_{*-} D_{NN}(\mathbf{u}_-) - \pi_-)|_- - (\mu_{*+} D_{NN}(\mathbf{u}_+) + (\nu_{*+} - \mu_{*+}) \operatorname{div} \mathbf{u}_+)|_+ - \sigma \Delta' H &= g_N, \\ \frac{1}{\rho_{*-}} (\mu_{*-} D_{NN}(\mathbf{u}_-) - \pi_-)|_- - \frac{1}{\rho_{*+}} (\mu_{*+} D_{NN}(\mathbf{u}_+) + (\nu_{*+} - \mu_{*+}) \operatorname{div} \mathbf{u}_+)|_+ &= g_{N+1}, \\ u_{-i}|_- - u_{+i}|_+ &= h_i \quad (i = 1, \dots, N-1), \\ \partial_t H - \left(\frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}} u_{-N} - \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}} u_{+N} \right) &= d, \end{aligned} \quad (1.5)$$

and the initial condition:

$$\mathbf{u}_\pm|_{t=0} = \mathbf{u}_{0\pm} \quad \text{in } \mathbb{R}_\pm^N, \quad H|_{t=0} = H_0 \quad \text{in } \mathbb{R}^N, \quad (1.6)$$

where we have set $\mathbf{u}_\pm = (u_{\pm 1}, \dots, u_{\pm N})$, $\mu_{*+} = \mu(\rho_{*+}, \theta_*)$, $\nu_{*+} = \nu_+(\rho_{*+}, \theta_*)$, $\mu_{*-} = \mu_-(\theta_*)$, $\Delta' H = \sum_{j=1}^{N-1} \partial_j^2 H$, $S_{*+}(\mathbf{u}_+) = \mu_{*+} \mathbf{D}(\mathbf{u}_+) + (\nu_{*+} - \mu_{*+}) \operatorname{div} \mathbf{u}_+ \mathbf{I}$, and $S_{*-}(\mathbf{u}_-) = \mu_{*-} \mathbf{D}(\mathbf{u}_-)$.

And, another is the interface problem for the heat equations:

$$\begin{aligned} \rho_{*+} \kappa_{*+} \partial_t \theta_+ - d_{*+} \Delta \theta_+ &= \tilde{f}_+ \quad \text{in } \mathbb{R}_+^N \times (0, T), \\ \rho_{*-} \kappa_{*-} \partial_t \theta_- - d_{*-} \Delta \theta_- &= \tilde{f}_- \quad \text{in } \mathbb{R}_-^N \times (0, T), \end{aligned} \quad (1.7)$$

subject to the interface condition: for $x \in \mathbb{R}_0^N$ and $t \in (0, T)$

$$\theta_-|_- - \theta_+|_+ = 0, \quad d_{*+} \partial_N \theta_-|_- - d_{*+} \partial_N \theta_+|_+ = \tilde{g}, \quad (1.8)$$

and the initial condition:

$$\theta_\pm|_{t=0} = \theta_{0\pm} \quad \text{on } \mathbb{R}_\pm^N, \quad (1.9)$$

where we have set $d_{*+} = d(\rho_{*+}, \theta_*)$, $\kappa_{*+} = \kappa(\rho_{*+}, \theta_*)$, $d_{*-} = d_-(\theta_*)$ and $\kappa_{*-} = \kappa_-(\theta_*)$. Note that the interface condition (1.5) is equivalent to the following interface condition:

$$\begin{aligned} \mu_{*-} D_{iN}(\mathbf{u}_-)|_- - \mu_{*+} D_{iN}(\mathbf{u}_+)|_+ &= g_i \quad (i = 1, \dots, N-1), \\ (\mu_{*-} D_{NN}(\mathbf{u}_-) - \pi_-)|_- &= \frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}} (\sigma \Delta' H + g_N - \rho_{*+} g_{N+1}), \\ (\mu_{*+} D_{NN}(\mathbf{u}_+) + (\nu_{*+} - \mu_{*+}) \operatorname{div} \mathbf{u}_+)|_+ &= \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}} (\sigma \Delta' H + g_N - \rho_{*-} g_{N+1}), \\ u_{-i}|_- - u_{+i}|_+ &= h_i \quad (i = 1, \dots, N-1), \\ \partial_t H - \left(\frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}} u_{-N} - \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}} u_{+N} \right) &= d \end{aligned} \quad (1.10)$$

The purpose of this paper is to prove the following theorem about the maximal L_p - L_q regularity for problem (1.4), (1.5), (1.6).

Theorem 1.1. *Let $1 < p, q < \infty$ and $0 < T < \infty$. Assume that $\rho_{*-} \neq \rho_{*+}$. Then, given right-hand sides of (1.4) and (1.5)*

$$\begin{aligned} \mathbf{f}_\pm &\in L_p((0, T), L_q(\mathbb{R}_\pm^N)^N), \quad f_{\operatorname{div}} \in L_p((0, T), W_q^1(\mathbb{R}_-^N)) \cap W_p^1((0, T), W_q^{-1}(\mathbb{R}_-^N)) \\ \mathbf{f}_{\operatorname{div}} &\in W_p^1((0, T), L_q(\mathbb{R}_-^N)^N), \quad g_i \in L_p((0, T), W_q^1(\mathbb{R}_-^N)) \cap W_p^1((0, T), W_q^{-1}(\mathbb{R}_-^N)) \quad (i = 1, \dots, N+1), \\ h_j &\in L_p((0, T), W_q^2(\mathbb{R}_-^N)) \cap W_p^1((0, T), L_p(\mathbb{R}_-^N)) \quad (j = 1, \dots, N-1), \quad d \in L_p((0, T), W_q^2(\mathbb{R}_-^N)), \end{aligned}$$

and initial data $\mathbf{u}_{0\pm} \in B_{q,p}^{2(1-1/p)}(\mathbb{R}_\pm^N)^N$ and $H_0 \in B_{q,p}^{3-1/p}(\mathbb{R}^N)$ satisfying the compatibility conditions:

$$\begin{aligned}
\operatorname{div} \mathbf{u}_{0-} &= f_-|_{t=0}, \quad \mathbf{u}_{0-} = \mathbf{f}_{\operatorname{div}}|_{t=0} && \text{in } \mathbb{R}_-^N, \\
\mu_{*-} D_{iN}(\mathbf{u}_{0-})|_- - \mu_{*+} D_{iN}(\mathbf{u}_{0+})|_+ &= g_i|_{t=0} \quad (i = 1, \dots, N-1) && \text{on } \mathbb{R}_0^N, \\
(\mu_{*+} D_{NN}(\hat{\mathbf{u}}_{0+}) + (\nu_{*+} - \mu_{*+}) \operatorname{div} \hat{\mathbf{u}}_{0+})|_+ &&& \\
&= \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}} (\sigma \Delta' H_0 + g_N|_{t=0} - \rho_{*-} g_{N+1}|_{t=0}) && \text{on } \mathbb{R}_0^N, \\
u_{0-i}|_- - u_{0+i}|_+ &= h_i|_{t=0} \quad (i = 1, \dots, N-1) && \text{on } \mathbb{R}_0^N.
\end{aligned} \tag{1.11}$$

then, problem (1.4), (1.5), (1.6) admits unique solutions \mathbf{u}_\pm , π_- and H with

$$\begin{aligned}
\mathbf{u}_\pm &\in L_p((0, T), W_q^2(\mathbb{R}_\pm^N)^N) \cap W_p^1((0, T), L_q(\mathbb{R}_\pm^N)^N), \\
\pi &\in L_p((0, T), \hat{W}_q^1(\mathbb{R}^N)), \\
H &\in L_p((0, T), W_q^3(\mathbb{R}^N)) \cap W_p^1((0, T), W_q^2(\mathbb{R}^N))
\end{aligned}$$

possessing the estimates: $\mathcal{I}_{p,q}(\mathbf{u}_\pm, \pi_-, H)(t) \leq C e^{\gamma t} \mathbb{F}_{p,q}(t)$ for any $t \in (0, T)$ with some positive constants C and γ independent of t and T , where we have set

$$\begin{aligned}
\mathcal{I}_{p,q}(\mathbf{u}_\pm, \pi_-, H)(t) &= \|\mathbf{u}_+\|_{L_p((0,t), W_q^2(\mathbb{R}_+^N))} + \|\partial_t \mathbf{u}_+\|_{L_p((0,t), L_q(\mathbb{R}_+^N))} + \|\mathbf{u}_-\|_{L_p((0,t), W_q^2(\mathbb{R}_-^N))} + \|\partial_t \mathbf{u}_-\|_{L_p((0,t), L_q(\mathbb{R}_-^N))} \\
&+ \|\nabla \pi_-\|_{L_p((0,t), L_q(\mathbb{R}_-^N))} + \|H\|_{L_p((0,t), W_q^3(\mathbb{R}^N))} + \|\partial_t H\|_{L_p((0,t), W_q^2(\mathbb{R}^N))}, \\
\mathbb{F}_{p,q}(t) &= \left\{ \sum_{\ell=\pm} (\|\mathbf{u}_{0\ell}\|_{B_{q,p}^{2(1-1/p)}(\mathbb{R}_\ell^N)} + \|\mathbf{f}_\ell\|_{L_p((0,t), L_q(\mathbb{R}_\ell^N))}) + \|f_{\operatorname{div}}\|_{L_p((0,t), W_q^1(\mathbb{R}_-^N))} + \|\partial_t f_{\operatorname{div}}\|_{L_p((0,t), W_q^{-1}(\mathbb{R}_-^N))} \right. \\
&+ \|\mathbf{f}_{\operatorname{div}}\|_{L_p((0,T), L_q(\mathbb{R}_-^N))} + \sum_{i=1}^{N+1} (\|g_i\|_{L_p((0,t), W_q^1(\mathbb{R}^N))} + \|\partial_t g_i\|_{L_p((0,t), W_q^{-1}(\mathbb{R}^N))}) \\
&\left. + \sum_{j=1}^{N-1} (\|h_j\|_{L_p((0,t), W_q^2(\mathbb{R}^N))} + \|\partial_t h_j\|_{L_p((0,t), L_q(\mathbb{R}^N))}) + \|d\|_{L_p((0,t), W_q^2(\mathbb{R}^N))} \right\}.
\end{aligned}$$

Remark 1.2. The maximal L_p - L_q regularity theorem for problem (1.7), (1.8) and (1.9) seems to be known and employing the similar argumentation to that in the proof of Theorem 1.1 given in the sequel we can prove it too, so that we do not consider problem (1.7), (1.8) and (1.9) in this paper.

Remark 1.3. (1) The mathematical study of the compressible and incompressible two phase problem is quite rare as far as the author knows. First Denisova [1] studied the evolution of the compressible and incompressible two phase flow with sharp interface without phase transition under some restriction on the viscosity coefficients. Recently, Kubo, Shibata and Soga [4] studied the same problem as in [1] without surface tension and phase transition and proved the maximal L_p - L_q regularity under the assumption that viscosity coefficients are positive constants. The derivation and the local well-posedness of problem (1.1), (1.2) and (1.3) are treated in Shibata [10] and all the results of this paper and in [10] were announced in the abstract of 39th Sapporo symposium on PDE at Hokkaido University (cf. [11]). The incompressible and incompressible two phase problem with phase transition was studied by J. Prüss, et al. [6, 7, 8].

Notation Here, we summarize our symbols used throughout the paper. \mathbb{N} , \mathbb{R} and \mathbb{C} denote the sets of all natural numbers, real numbers and complex numbers, respectively. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any multi-index $\kappa = (\kappa_1, \dots, \kappa_N) \in \mathbb{N}_0^N$, we write $|\kappa| = \kappa_1 + \dots + \kappa_N$ and $\partial_x^\kappa = \partial_1^{\kappa_1} \dots \partial_N^{\kappa_N}$ with $x = (x_1, \dots, x_N)$ and $\partial_j = \partial/\partial x_j$. For any scalar function f and N -vector of functions $\mathbf{g} = (g_1, \dots, g_N)$, we set

$$\begin{aligned}
\nabla f &= (\partial_1 f, \dots, \partial_N f), \quad \nabla \mathbf{g} = (\partial_i g_j \mid i, j = 1, \dots, N), \\
\nabla^2 f &= (\partial^\alpha f \mid |\alpha| = 2), \quad \nabla^2 \mathbf{g} = (\partial^\alpha g_i \mid |\alpha| = 2, i = 1, \dots, N).
\end{aligned}$$

We use bold small letters to denote N -vector or N -vector valued functions and bold capital letters to denote $N \times N$ matrix or $N \times N$ matrix valued functions, respectively. For any domain D and $1 \leq q \leq \infty$, $L_q(D)$, $W_q^m(D)$ and $B_{q,p}^\ell(D)$ denote the standard Lebesgue space, Sobolev space and Besov space, while $\|\cdot\|_{L_q(D)}$, $\|\cdot\|_{W_q^m(D)}$ and $\|\cdot\|_{B_{q,p}^\ell(D)}$ denote their norms, respectively. We set $W_q^0(D) = L_q(D)$. $\dot{W}_q^1(D)$ is a homogeneous space defined by $\dot{W}_q^1(D) = \{f \in L_{q,\text{loc}}(D) \mid \nabla f \in L_q(D)\}$. $W_q^{-1}(\mathbb{R}^N)$ denotes the usual Bessel potential space of order -1 on \mathbb{R}^N and $W_q^{-1}(\mathbb{R}_\pm^N) = \{f \in L_{1,\text{loc}}(\mathbb{R}_\pm^N) \mid f = g \text{ in } \mathbb{R}_\pm^N \text{ for some } g \in W_q^{-1}(\mathbb{R}^N)\}$. For any Banach space X with norm $\|\cdot\|_X$ and $1 \leq p \leq \infty$, $L_p((a,b), X)$ and $W_p^m((a,b), X)$ denote the usual Lebesgue space and Sobolev space of X -valued functions defined on an interval (a,b) , while $\|\cdot\|_{L_p((a,b), X)}$ and $\|\cdot\|_{W_p^m((a,b), X)}$ denote their norms, respectively. For any $\gamma \in \mathbb{R}$ we set $\|e^{\gamma t} f\|_{L_p((a,b), X)} = \left(\int_a^b (e^{\gamma t} \|f(t)\|_X)^p dt \right)^{1/p}$. The d -product space of X is defined by $X^d = \{f = (f_1, \dots, f_d) \mid f_i \in X \text{ } (i = 1, \dots, d)\}$, while its norm is denoted by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$ for the sake of simplicity. For any two Banach spaces X and Y , $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y . $\text{Hol}(U, X)$ denotes the set of all X -valued holomorphic functions defined on U . For $\mathbf{a} = (a_1, \dots, a_N)$ and $\mathbf{b} = (b_1, \dots, b_N) \in \mathbb{R}^N$, we set $\mathbf{a} \cdot \mathbf{b} = \langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^N a_j b_j$. For scalar functions f, g and N -vectors of functions \mathbf{f}, \mathbf{g} , we set $(f, g)_D = \int_D f(x)g(x) dx$ and $(\mathbf{f}, \mathbf{g})_D = \int_D \mathbf{f}(x) \cdot \mathbf{g}(x) dx$. The letter C denotes generic constants and the constant $C_{a,b,\dots}$ depends on a, b, \dots . The values of constants C and $C_{a,b,\dots}$ may change from line to line.

The paper is organized as follows. In Sect.2, we state the main results for the \mathcal{R} bounded solution operators to the corresponding resolvent problem to time dependent problem (1.4), (1.10) and (1.6). From Sect.3 through Sect.5, we consider the problem without surface tension. In Sect.3, we give an exact solution formulas to the resolvent problem. In Sect.4, we give some estimates for the multipliers appearing in the solution formula. In Sect.5 we analyze the Lopatinski determinant. In Sect.6, we prove the main result for the \mathcal{R} bounded solution operators. In Sect.7, using the \mathcal{R} bounded solution operator, we prove Theorem 1.1.

2 Main results for the \mathcal{R} bounded solution operators

In the sequel, for notational simplicity $\rho_{*\pm}$, $\mu_{*\pm}$ and ν_{*+} are denoted by ρ_\pm , μ_\pm and ν_+ , respectively. And, $\mathbf{S}_\pm(\mathbf{u}_\pm)$ are redefined by

$$\mathbf{S}_+(\mathbf{u}_+) = \mu_+ \mathbf{D}(\mathbf{u}_+) + (\nu_+ - \mu_+) \text{div } \mathbf{u}_+ \mathbf{I}, \quad \mathbf{S}_-(\mathbf{u}_-) = \mu_- \mathbf{D}(\mathbf{u}_-).$$

To prove Theorem 1.1, we consider the following generalized resolvent problem:

$$\begin{aligned} \rho_+ \lambda \mathbf{u}_+ - \text{Div } \mathbf{S}_+(\mathbf{u}_+) &= \mathbf{f}_+ && \text{in } \mathbb{R}_+^N, \\ \rho_- \lambda \mathbf{u}_- - \text{Div } \mathbf{S}_-(\mathbf{u}_-) + \nabla \pi_- &= \mathbf{f}_-, \quad \text{div } \mathbf{u}_- = \tilde{f}_- = \text{div } \tilde{\mathbf{f}}_- && \text{in } \mathbb{R}_-^N, \\ \mu_- \mathbf{D}_{mN}(\mathbf{u}_-)|_- - \mu_+ \mathbf{D}_{mN}(\mathbf{u}_+)|_+ &= g_m, \\ (\mu_- \mathbf{D}_{NN}(\mathbf{u}_-) - \pi_-)|_- &= \frac{\rho_- \sigma}{\rho_- - \rho_+} \Delta' H + g_N, \\ (\mu_+ \mathbf{D}_{NN}(\mathbf{u}_+) + (\nu_+ - \mu_+) \text{div } \mathbf{u}_+)|_+ &= \frac{\rho_+ \sigma}{\rho_- - \rho_+} \Delta' H + g_{N+1}, \\ u_{-m}|_- - u_{+m}|_+ &= h_m, \\ \lambda H - \left(\frac{\rho_-}{\rho_- - \rho_+} u_{-N}|_- - \frac{\rho_+}{\rho_- - \rho_+} u_{+N}|_+ \right) &= d, \end{aligned} \tag{2.1}$$

which is corresponding to the time dependent problem (1.4), (1.10) and (1.6). Here and in the sequel, m runs from 1 through $N-1$.

Before stating the main result of this section, we first introduce the definition of \mathcal{R} -boundedness.

Definition 2.1. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called \mathcal{R} -bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and sequences

$\{r_j(u)\}_{j=1}^n$ of independent, symmetric, $\{-1, 1\}$ -valued random variables on $[0, 1]$ there holds the inequality:

$$\left\{ \int_0^1 \left\| \sum_{j=1}^n r_j(u) T_j f_j \right\|_Y^p du \right\}^{\frac{1}{p}} \leq C \left\{ \int_0^1 \left\| \sum_{j=1}^n r_j(u) f_j \right\|_X^p du \right\}^{\frac{1}{p}}.$$

The smallest such C is called \mathcal{R} -bound of \mathcal{T} , which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$. Here and in the following, $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y .

The following theorem is a main result for problem (2.1).

Theorem 2.2. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Set*

$$\begin{aligned} \Sigma_\epsilon &= \{\lambda = \gamma + i\tau \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon\}, \quad \Sigma_{\epsilon, \lambda_0} = \{\lambda \in \Sigma_\epsilon \mid |\lambda| > \lambda_0\} \quad (\lambda_0 \geq 0), \\ X_q &= \{(\mathbf{f}_+, \mathbf{f}_-, f_{\text{div}}, \mathbf{f}_{\text{div}}, \mathbf{g}, \mathbf{h}, d) \mid \mathbf{f}_+ \in L_q(\mathbb{R}_+^N)^N, \mathbf{f}_-, \mathbf{f}_{\text{div}} \in L_q(\mathbb{R}_-^N)^N, f_{\text{div}} \in W_q^1(\mathbb{R}^N), \\ &\quad \mathbf{g} = (g_1, \dots, g_{N+1}) \in W_q^1(\mathbb{R}^N)^{N+1}, \mathbf{h} = (h_1, \dots, h_{N-1}) \in W_q^2(\mathbb{R}^N)^{N-1}, d \in W_q^2(\mathbb{R}^N)\}, \\ \mathcal{X}_q &= \{F = (F_{+1}, F_{-1}, F_{-2}, F_{-3}, F_{-4}, F_1, F_2, F_3, F_4, F_5, F_6) \mid F_{\pm 1} \in L_q(\mathbb{R}_\pm^N), \\ &\quad F_{-2} \in L_q(\mathbb{R}_-^N), F_{-3}, F_{-4} \in L_q(\mathbb{R}_-^N)^N, F_1 \in L_q(\mathbb{R}^N)^{N+1}, F_2 \in L_q(\mathbb{R}^N)^{(N+1)N}, \\ &\quad F_3 \in L_q(\mathbb{R}^N)^{N-1}, F_4 \in L_q(\mathbb{R}^N)^{(N-1)N}, F_5 \in L_q(\mathbb{R}^N)^{(N-1)N^2}, F_6 \in W_q^2(\mathbb{R}^N)\}. \end{aligned}$$

Then, there exist a constant $\lambda_0 > 0$ and operator families $\mathcal{A}_\pm(\lambda) \in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(X_q, W_q^2(\mathbb{R}_\pm^N)^N))$, $\mathcal{P}_- \in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(X_q, \hat{W}_q^1(\mathbb{R}^N)))$, and $\mathcal{H}(\lambda) \in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(X_q, W_q^3(\mathbb{R}^N)))$ such that for any $\lambda \in \Sigma_{\epsilon, \lambda_0}$ and $\mathbf{F} = (\mathbf{f}_+, \mathbf{f}_-, f_{\text{div}}, \mathbf{f}_{\text{div}}, \mathbf{g}, \mathbf{h}, d) \in X_q$, $\mathbf{u}_\pm = \mathcal{A}_\pm(\lambda)\mathbf{F}_\lambda$, $\pi_- = \mathcal{P}_-(\lambda)\mathbf{F}_\lambda$ and $H = \mathcal{H}(\lambda)\mathbf{F}_\lambda$ are unique solutions of problem (2.1) and we have

$$\begin{aligned} \mathcal{R}_{\mathcal{L}(X_q, L_q(\mathbb{R}_\pm^N)^{N+N^2+N^3})}(\{(\tau \partial_\tau)^\ell G_\lambda^1 \mathcal{A}_\pm(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq c \quad (\ell = 0, 1), \\ \mathcal{R}_{\mathcal{L}(X_q, L_q(\mathbb{R}_-^N)^N)}(\{(\tau \partial_\tau)^\ell \nabla \mathcal{P}_-(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq c \quad (\ell = 0, 1), \\ \mathcal{R}_{\mathcal{L}(X_q, W_q^2(\mathbb{R}^N)^{N+1})}(\{(\tau \partial_\tau)^\ell G_\lambda^2 \mathcal{H}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq c \quad (\ell = 0, 1) \end{aligned}$$

with some constant c . Here, $G_\lambda^1 \mathcal{A}_\pm(\lambda) = (\lambda \mathcal{A}_\pm(\lambda), \lambda^{1/2} \nabla \mathcal{A}_\pm(\lambda), \nabla^2 \mathcal{A}_\pm(\lambda))$, $G_\lambda^2 \mathcal{H}(\lambda) = (\lambda \mathcal{H}(\lambda), \nabla \mathcal{H}(\lambda))$, and $\mathbf{F}_\lambda = (F_+, F_-, \lambda^{1/2} f_{\text{div}}, \nabla f_{\text{div}}, \lambda \mathbf{f}_{\text{div}}, \lambda^{1/2} \mathbf{g}, \nabla \mathbf{g}, \lambda \mathbf{h}, \lambda^{1/2} \nabla \mathbf{h}, \nabla^2 \mathbf{h}, d)$.

Remark 2.3. $F_\pm, F_{-2}, F_{-3}, F_{-4}, F_1, F_2, F_3, F_4, F_5$ and F_6 are corresponding variables to $\mathbf{f}_\pm, \lambda^{1/2} f_{\text{div}}, \nabla f_{\text{div}}, \lambda \mathbf{f}_{\text{div}}, \lambda^{1/2} \mathbf{g}, \nabla \mathbf{g}, \lambda \mathbf{h}, \lambda^{1/2} \nabla \mathbf{h}, \nabla^2 \mathbf{h}$ and d , respectively.

To prove Theorem 2.2, as auxiliary problem, we consider the following two equations.

$$\begin{cases} \rho_+ \lambda \mathbf{u}_+ - \text{Div } \mathbf{S}_+(\mathbf{u}_+) = \mathbf{f}_+ & \text{in } \mathbb{R}_+^N, \\ \mu_+ \mathbf{D}_{mN}(\mathbf{u}_+)|_+ = 0, \quad (\mu_+ \mathbf{D}_{NN}(\mathbf{u}_+) + (\nu_+ - \mu_+) \text{div } \mathbf{u}_+)|_+ = g_{N+1} \end{cases} \quad (2.2)$$

and

$$\begin{cases} \rho_- \lambda \mathbf{u}_- - \text{Div } \mathbf{S}_-(\mathbf{u}_-) + \nabla \pi_- = \mathbf{f}_-, \quad \text{div } \mathbf{u}_- = \tilde{f}_- = \text{div } \tilde{\mathbf{f}}_- & \text{in } \mathbb{R}_-^N, \\ \mu_- \mathbf{D}_{mN}(\mathbf{u}_-)|_- = g_m, \quad (\mu_- \mathbf{D}_{NN}(\mathbf{u}_-) - \pi_-)|_- = g_N. \end{cases} \quad (2.3)$$

The existence of \mathcal{R} bounded solution operators of (2.2) and (2.3) were proved in Shibata [9] (cf also [14]) and Götze and Shibata [3], respectively. In fact, we know the following two theorems.

Theorem 2.4 ([3]). *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and $\lambda_0 > 0$. Set*

$$\begin{aligned} Y_{q+} &= \{(\mathbf{f}_+, g) \mid \mathbf{f}_+ \in L_q(\mathbb{R}_+^N)^N, g_{N+1} \in W_q^1(\mathbb{R}_+^N)\} \\ \mathcal{Y}_{q+} &= \{F = (F_{+1}, \tilde{F}_{+1}, \tilde{F}_{+2}) \mid F_{+1} \in L_q(\mathbb{R}_+^N)^N, \tilde{F}_{+1} \in L_q(\mathbb{R}_+^N), \tilde{F}_{+2} \in L_q(\mathbb{R}_+^N)^N\}. \end{aligned}$$

Then, there exist an operator family $\mathcal{A}_{+1}(\lambda) \in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(\mathcal{Y}_{q+}, W_q^2(\mathbb{R}_+^N)^N))$ such that for any $\lambda \in \Sigma_{\epsilon, \lambda_0}$ and $(\mathbf{f}_+, g_{N+1}) \in Y_{q+}$, $\mathbf{u}_+ = \mathcal{A}_{+1}(\lambda)(\mathbf{f}_+, \lambda^{1/2}g_{N+1}, \nabla g_{N+1})$ is a unique solution of problem (2.2) and we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_{q+}, L_q(\mathbb{R}_+^N)^{N+N^2+N^3})}(\{(\tau \partial_\tau)^\ell G_\lambda^1 \mathcal{A}_{+1}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) \leq c \quad (\ell = 0, 1)$$

with some constant c .

Remark 2.5. \tilde{F}_{+1} and \tilde{F}_{+2} are corresponding variables to $\lambda^{1/2}g_{N+1}$ and ∇g_{N+1} , respectively.

Theorem 2.6 ([9]). Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Set

$$\begin{aligned} Y_{q-} &= \{(\mathbf{f}_-, f_{\text{div}}, \mathbf{f}_{\text{div}}, \tilde{\mathbf{g}}) \mid \mathbf{f}_-, \mathbf{f}_{\text{div}} \in L_q(\mathbb{R}_-^N), f_{\text{div}} \in W_q^1(\mathbb{R}_-^N), \tilde{\mathbf{g}} = (g_1, \dots, g_N) \in W_q^1(\mathbb{R}_-^N)^N\}, \\ \mathcal{Y}_{q-} &= \{F = (F_{-1}, F_{-2}, F_{-3}, F_{-4}, \tilde{F}_{-1}, \tilde{F}_{-2}) \mid F_{-1} \in L_q(\mathbb{R}_-^N)^N, \\ &\quad F_{-2} \in L_q(\mathbb{R}_-^N), F_{-3}, F_{-4} \in L_q(\mathbb{R}_-^N)^N, \tilde{F}_{-1} \in L_q(\mathbb{R}_-^N)^N, \tilde{F}_{-2} \in L_q(\mathbb{R}_-^N)^{N^2}\}. \end{aligned}$$

Then, there exist operator families $\mathcal{A}_{-1}(\lambda)$ and $\mathcal{P}_{-1}(\lambda)$ with

$$\mathcal{A}_{-1}(\lambda) \in \text{Hol}(\Sigma_\epsilon, \mathcal{L}(\mathcal{Y}_{q-}, W_q^2(\mathbb{R}_-^N)^N)), \quad \mathcal{P}_{-1}(\lambda) \in \text{Hol}(\Sigma_\epsilon, \mathcal{L}(\mathcal{Y}_{q-}, \hat{W}_q^1(\mathbb{R}_-^N)))$$

such that for any $\lambda \in \Sigma_\epsilon$ and $\mathbf{F} = (\mathbf{f}_-, f_{\text{div}}, \mathbf{f}_{\text{div}}, \tilde{\mathbf{g}}) \in Y_{q-}$, $\mathbf{u}_- = \mathcal{A}_{-1}(\lambda)\tilde{\mathbf{F}}_\lambda$ and $\pi_- = \mathcal{P}_{-1}(\lambda)\tilde{\mathbf{F}}_\lambda$ are unique solutions of problem (2.3) and we have

$$\begin{aligned} \mathcal{R}_{\mathcal{L}(\mathcal{Y}_{q-}, L_q(\mathbb{R}_-^N)^{N+N^2+N^3})}(\{(\tau \partial_\tau)^\ell G_\lambda^1 \mathcal{A}_{-1}(\lambda) \mid \lambda \in \Sigma_\epsilon\}) &\leq c \quad (\ell = 0, 1), \\ \mathcal{R}_{\mathcal{L}(\mathcal{Y}_{q-}, L_q(\mathbb{R}_-^N)^N)}(\{(\tau \partial_\tau)^\ell \nabla \mathcal{P}_{-1}(\lambda) \mid \lambda \in \Sigma_\epsilon\}) &\leq c \quad (\ell = 0, 1) \end{aligned}$$

with some constant c . Here, $\tilde{\mathbf{F}}_\lambda = (\mathbf{f}_-, \lambda^{1/2}f_{\text{div}}, \nabla f_{\text{div}}, \lambda \mathbf{f}_{\text{div}}, \lambda^{1/2}\tilde{\mathbf{g}}, \nabla \tilde{\mathbf{g}})$.

Remark 2.7. \tilde{F}_{-1} and \tilde{F}_{-2} are corresponding variables to $\lambda^{1/2}\tilde{\mathbf{g}}$ and $\nabla \tilde{\mathbf{g}}$, respectively.

Thus, it is sufficient to consider problem (2.1) with $\mathbf{f}_\pm = 0$, $f_{\text{div}} = 0$, $\mathbf{f}_{\text{div}} = 0$ and $g_j = 0$ for $j = 1, \dots, N+1$. Finally, we consider one more auxiliary problem:

$$\begin{aligned} \rho_+ \lambda \mathbf{u}_+ - \text{Div } \mathbf{S}_+(\mathbf{u}_+) &= 0 && \text{in } \mathbb{R}_+^N, \\ \rho_- \lambda \mathbf{u}_- - \text{Div } \mathbf{S}_-(\mathbf{u}_-) + \nabla \pi_- &= 0, \quad \text{div } \mathbf{u}_- = 0 && \text{in } \mathbb{R}_-^N, \\ \mu_- \mathbf{D}_{mN}(\mathbf{u}_-)|_- - \mu_+ \mathbf{D}_{mN}(\mathbf{u}_+)|_+ &= 0, \\ (\mu_- D_{NN}(\mathbf{u}_-) - \pi_-)|_- &= 0, \\ (\mu_+ \mathbf{D}_{NN}(\mathbf{u}_+) + (\nu_+ - \mu_+) \text{div } \mathbf{u}_+)|_+ &= 0, \\ u_{-m}|_- - u_{+m}|_+ &= h_m, \end{aligned} \tag{2.4}$$

From Sect.3 through Sect.5, we prove the following theorem.

Theorem 2.8. Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Set

$$\begin{aligned} Z_q &= \{\mathbf{h} = (h_1, \dots, h_{N-1}) \in W_q^1(\mathbb{R}^N)^{N-1}\}, \\ \mathcal{Z}_q &= \{F = (F_3, F_4, F_5) \mid F_3 \in L_q(\mathbb{R}^N)^{N-1}, F_4 \in L_q(\mathbb{R}^N)^{(N-1)N}, F_5 \in L_q(\mathbb{R}^N)^{(N-1)N^2}\}. \end{aligned}$$

Then, there exist operator families $\mathcal{A}_{\pm 2}(\lambda)$ and $\mathcal{P}_{-2}(\lambda)$ with

$$\mathcal{A}_{\pm 2}(\lambda) \in \text{Hol}(\Sigma_\epsilon \mathcal{L}(\mathcal{Z}_q, W_q^2(\mathbb{R}_\pm^N)^N)), \quad \mathcal{P}_{-2}(\lambda) \in \text{Hol}(\Sigma_\epsilon, \mathcal{L}(\mathcal{Z}_q, \hat{W}_q^1(\mathbb{R}_-^N)))$$

such that for any $\lambda \in \Sigma_\epsilon$ and $(\mathbf{h}, H) \in Z_q$, $\mathbf{u}_\pm = \mathcal{A}_{\pm 2}(\lambda)\mathbf{H}_\lambda$ and $\pi_- = \mathcal{P}_{-2}(\lambda)\mathbf{H}_\lambda$ are unique solutions of problem (2.4) and we have

$$\begin{aligned} \mathcal{R}_{\mathcal{L}(\mathcal{Z}_q, L_q(\mathbb{R}_\pm^N)^{N+N^2+N^3})}(\{(\tau \partial_\tau)^\ell G_\lambda^1 \mathcal{A}_{\pm 2}(\lambda) \mid \lambda \in \Sigma_\epsilon\}) &\leq c \quad (\ell = 0, 1), \\ \mathcal{R}_{\mathcal{L}(\mathcal{Z}_q, L_q(\mathbb{R}_-^N)^N)}(\{(\tau \partial_\tau)^\ell \nabla \mathcal{P}_{-2}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq c \quad (\ell = 0, 1) \end{aligned}$$

with some constant c . Here, $\mathbf{H}_\lambda = (\lambda \mathbf{h}, \lambda^{1/2} \nabla \mathbf{h}, \nabla^2 \mathbf{h})$.

3 Solution formulas for problem (2.4)

In this section, we consider the following equations:

$$\begin{aligned}
\rho_+ \lambda \mathbf{u}_+ - \operatorname{Div} \mathbf{S}_+(\mathbf{u}_+) &= 0 & \text{in } \mathbb{R}_+^N, \\
\rho_- \lambda \mathbf{u}_- - \operatorname{Div} \mathbf{S}_-(\mathbf{u}_-) + \nabla \pi_- &= 0, \quad \operatorname{div} \mathbf{u}_- = 0 & \text{in } \mathbb{R}_-^N, \\
\mu_- \mathbf{D}_{mN}(\mathbf{u}_-)|_- - \mu_+ \mathbf{D}_{mN}(\mathbf{u}_+)|_+ &= 0, \\
(\mu_- D_{NN}(\mathbf{u}_-) - \pi_-)|_- &= \sigma_- \Delta' H, \\
(\mu_+ \mathbf{D}_{NN}(\mathbf{u}_+) + (\nu_+ - \mu_+) \operatorname{div} \mathbf{u}_+)|_+ &= \sigma_+ \Delta' H, \\
u_{-m}|_- - u_{+m}|_+ &= h_m.
\end{aligned} \tag{3.1}$$

where, we have added $\sigma_\pm \Delta' H$ with $\sigma_\pm = \frac{\rho_\pm \sigma}{\rho_- - \rho_+}$ to (2.4) for the latter use. Let $\hat{v} = \mathcal{F}_{x'}[v](\xi', x_N)$ denote the partial Fourier transform with respect to the tangential variable $x' = (x_1, \dots, x_{N-1})$ with $\xi' = (\xi_1, \dots, \xi_{N-1})$ defined by $\mathcal{F}_{x'}[v](\xi', x_N) = \int_{\mathbb{R}^{N-1}} e^{-ix' \cdot \xi'} v(x', x_N) dx'$. Using the formulas:

$$\operatorname{Div} \mathbf{S}_+(\mathbf{u}_+) = \mu_+ \Delta \mathbf{u}_+ + \nu_+ \nabla \operatorname{div} \mathbf{u}_+, \quad \operatorname{Div} \mathbf{S}_-(\mathbf{u}_-) = \mu_- \Delta \mathbf{u}_-$$

and applying the partial Fourier transform to (3.1), we transfer problem (3.1) to the ordinary differential equations:

$$\begin{cases}
\rho_+ \lambda \hat{u}_{+j} + \mu_+ |\xi'|^2 \hat{u}_+ - \mu_+ D_N^2 \hat{u}_{+j} - \nu_+ i \xi_j (\hat{u}'_+ + D_N \hat{u}_{+N}) = 0 & \text{for } x_N > 0, \\
\rho_+ \lambda \hat{u}_{+N} + \mu_+ |\xi'|^2 \hat{u}_{+N} - \mu_+ D_N^2 \hat{u}_{+N} - \nu_+ D_N (i \xi' \cdot \hat{u}'_+ + D_N \hat{u}_{+N}) = 0 & \text{for } x_N > 0, \\
\rho_- \lambda \hat{u}_{-j} + \mu_- |\xi'|^2 \hat{u}_{-j} - \mu_- D_N^2 \hat{u}_{-j} + i \xi_j \hat{\pi}_- = 0 & \text{for } x_N < 0, \\
\rho_- \lambda \hat{u}_{-N} + \mu_- |\xi'|^2 \hat{u}_{-N} - \mu_- D_N^2 \hat{u}_{-N} + D_N \hat{\pi}_- = 0 & \text{for } x_N < 0, \\
i \xi' \cdot \hat{u}'_- + D_N \hat{u}_{-N} = 0 & \text{for } x_N < 0,
\end{cases} \tag{3.2}$$

subject to the interface condition:

$$\begin{cases}
\mu_- (D_N \hat{u}_{-m} + i \xi_m \hat{u}_{-N})|_- - \mu_+ (D_N \hat{u}_{-m} + i \xi_m \hat{u}_{-N})|_+ = 0, \\
(2\mu_- D_N \hat{u}_{-N} - \hat{\pi}_-)|_- = -\sigma_- A^2 \hat{H}(0), \\
(2\mu_+ D_N \hat{u}_{-N} + (\nu_+ - \mu_+) (i \xi' \cdot \hat{u}'_+ + D_N \hat{u}_{+N})|_+ = -\sigma_+ A^2 \hat{H}(0), \\
\hat{u}_{-m}|_- - \hat{u}_{+m}|_+ = \hat{h}_m(0)
\end{cases} \tag{3.3}$$

where $D_N = d/dx_N$ and $i \xi' \cdot \hat{v}' = \sum_{\ell=1}^{N-1} i \xi_\ell \hat{v}_\ell$ for $\mathbf{v} = (v_1, \dots, v_{N-1}, v_N)$. Here and in the sequel, j also runs from 1 through $N-1$. Applying the divergence to the first and second equations in (3.1), we have $\rho_+ \lambda \operatorname{div} \mathbf{u}_+ - (\mu_+ + \nu_+) \Delta \operatorname{div} \mathbf{u}_+ = 0$ in \mathbb{R}_+^N and $\Delta p_- = 0$ in \mathbb{R}_-^N , so that

$$(\rho_+ \lambda - (\mu_+ + \nu_+) \Delta)(\rho_+ \lambda - \mu_+ \Delta) \mathbf{u}_+ = 0 \quad \text{in } \mathbb{R}_+^N, \quad (\rho_- \lambda - \Delta) \Delta \mathbf{u}_- = 0 \quad \text{in } \mathbb{R}_-^N.$$

Thus, the characteristic roots of (3.2) are

$$A_+ = \sqrt{\rho_+ (\mu_+ + \nu_+)^{-1} \lambda + A^2}, \quad B_\pm = \sqrt{\rho_\pm (\mu_\pm)^{-1} \lambda + A^2}, \quad A = |\xi'|. \tag{3.4}$$

To state our solution formulas of problem: (3.2)- (3.3), we introduce some classes of multipliers.

Definition 3.1. Let $0 < \epsilon < \pi/2$, $\lambda_0 \geq 0$, and let s be a real number. Set

$$\tilde{\Sigma}_{\epsilon, \lambda_0} = \{(\lambda, \xi') \mid \lambda = \gamma + i\tau \in \Sigma_{\epsilon, \lambda_0}, \quad \xi' = (\xi_1, \dots, \xi_{N-1}) \in \mathbb{R}^{N-1} \setminus \{0\}\}.$$

Let $m(\lambda, \xi')$ be a function defined on $\tilde{\Sigma}_{\epsilon, \lambda_0}$.

- (1) $m(\lambda, \xi')$ is called a multiplier of order s with type 1 if for any multi-index $\kappa' = (\kappa_1, \dots, \kappa_{N-1}) \in \mathbb{N}_0^{N-1}$ and $(\lambda, \xi') \in \tilde{\Sigma}_{\epsilon, \lambda_0}$ there exists a constant $C_{\kappa'}$ depending on κ' , ϵ , μ_\pm , ν_+ and ρ_\pm such that there holds the estimates:

$$|\partial_{\xi'}^{\kappa'} m(\lambda, \xi')| \leq C_{\alpha'} (|\lambda|^{1/2} + A)^{s-|\kappa'|}, \quad |\partial_{\xi'}^{\kappa'} (\tau \frac{\partial m}{\partial \tau}(\lambda, \xi'))| \leq C_{\kappa'} (|\lambda|^{1/2} + A)^{s-|\kappa'|}. \tag{3.5}$$

- (2) $m(\lambda, \xi')$ is called a multiplier of order s with type 2 if for any multi-index $\kappa' = (\kappa_1, \dots, \kappa_{N-1}) \in \mathbb{N}_0^{N-1}$ and $(\lambda, \xi') \in \tilde{\Sigma}_{\epsilon, \lambda_0}$ there exists a constant $C_{\kappa'}$ depending on $\kappa', \epsilon, \mu_{\pm}, \nu_{\pm}$, and ρ_{\pm} such that there holds the estimates:

$$|\partial_{\xi'}^{\kappa'} m(\lambda, \xi')| \leq C_{\kappa'} (|\lambda|^{1/2} + A)^s A^{-|\kappa'|}, \quad |\partial_{\xi'}^{\kappa'} (\tau \frac{\partial m}{\partial \tau}(\lambda, \xi'))| \leq C_{\kappa'} (|\lambda|^{1/2} + A)^s A^{-|\kappa'|}. \quad (3.6)$$

Let $\mathbf{M}_{s,i}(\lambda_0)$ be the set of all multipliers of order s with type i ($i = 1, 2$).

Obviously, $\mathbf{M}_{s,i}(\lambda_0)$ are vector spaces on \mathbb{C} and for $0 \leq \lambda_0 < \lambda_1$, $\mathbf{M}_{s,i}(\lambda_0) \supset \mathbf{M}_{s,i}(\lambda_1)$. Moreover, by the fact: $||\lambda|^{1/2} + A|^{-|\alpha'|} \leq A^{-|\alpha'|}$ and the Leibniz rule, we have the following lemma immediately.

Lemma 3.2. *Let s_1, s_2 be two real numbers. Then, the following three assertions hold.*

- (1) Given $m_i \in \mathbf{M}_{s_i,1}(\lambda_0)$ ($i = 1, 2$), we have $m_1 m_2 \in \mathbf{M}_{s_1+s_2,1}(\lambda_0)$.
- (2) Given $\ell_i \in \mathbf{M}_{s_i,i}(\lambda_0)$ ($i = 1, 2$), we have $\ell_1 \ell_2 \in \mathbf{M}_{s_1+s_2,2}(\lambda_0)$.
- (3) Given $n_i \in \mathbf{M}_{s_i,2}(\lambda_0)$ ($i = 1, 2$), we have $n_1 n_2 \in \mathbf{M}_{s_1+s_2,2}(\lambda_0)$.

Remark 3.3. (1) We see easily that $i\xi_j \in \mathbf{M}_{1,1}(0)$ ($j = 1, \dots, N-1$), so that $A^2 \in \mathbf{M}_{2,1}(0)$. On the other hand, $A \in \mathbf{M}_{1,2}(0)$ and $A^{-1} \in \mathbf{M}_{-1,2}(0)$. Especially, $i\xi_j/A \in \mathbf{M}_{0,2}(0)$.

(2) $\mathbf{M}_{s,1}(\lambda_0) \subset \mathbf{M}_{s,2}(\lambda_0)$ for any $s \in \mathbb{R}$ and $\lambda_0 \geq 0$.

In this section, first of all we show the following solution formulas for problem (3.2)-(3.3):

$$\begin{aligned} \hat{u}_{+J} &= \sum_{k=1}^3 \hat{u}_{Jk}^+, \quad \hat{u}_{-J} = \sum_{k=1}^3 \hat{u}_{Jk}^-, \quad \hat{p}_- = e^{Ax_N} \left\{ \sum_{m=1}^{N-1} p_{m,1}^- \hat{h}_m(0) + A p_{N,1}^- \hat{H}(0) \right\}, \\ \hat{u}_{J1}^{\pm} &= A M_{\pm}(x_N) \left\{ \sum_{m=1}^{N-1} R_{Jm,0}^{\pm} \hat{h}_m(0) + A R_{JN,0}^{\pm} \hat{H}(0) \right\} \\ \hat{u}_{J2}^{\pm} &= A e^{\mp B_{\pm} x_N} \left\{ \sum_{m=1}^{N-1} S_{Jm,-1}^{\pm} \hat{h}_m(0) + A S_{JN,-1}^{\pm} \hat{H}(0) \right\} \\ \hat{u}_{J3}^{\pm} &= e^{\mp B_{\pm} x_N} T_{j,0}^{\pm} \hat{h}_j(0), \quad \hat{u}_{N,3}^{\pm} = 0 \end{aligned} \quad (3.7)$$

with

$$\begin{aligned} R_{Jm,0}^{\pm} &\in \mathbf{M}_{0,2}(0), \quad R_{JN,0}^{\pm} \in \mathbf{M}_{0,2}(0), \quad S_{Jm,-1}^{\pm} \in \mathbf{M}_{-1,2}(0), \quad S_{JN,-1}^{\pm} \in \mathbf{M}_{-1,2}(0) \\ T_{j,-1}^{\pm} &\in \mathbf{M}_{-1,1}(0), \quad T_{j,0}^{\pm} \in \mathbf{M}_{0,1}(0), \quad p_{m,1}^- \in \mathbf{M}_{1,2}(0), \quad p_{N,1}^- \in \mathbf{M}_{1,2}(0). \end{aligned} \quad (3.8)$$

Here and in the sequel, J runs from 1 through N . Recall that j and m run from 1 through $N-1$, respectively. Moreover, $M_{\pm}(x_N)$ denote the Stokes kernels defined by

$$M_+(x_N) = \frac{e^{-B_+ x_N} - e^{-A_+ x_N}}{B_+ - A_+}, \quad M_-(x_N) = \frac{e^{B_- x_N} - e^{A_- x_N}}{B_- - A_-}. \quad (3.9)$$

In the sequel, we prove (3.7). We look for solutions $\hat{u}_{\pm J}$ and \hat{p}_- to problem: (3.2)-(3.3) of the forms:

$$\begin{aligned} \hat{u}_{+J} &= \alpha_{+J} (e^{-B_+ x_N} - e^{-A_+ x_N}) + \beta_{+J} e^{-B_+ x_N}, \\ \hat{u}_{-J} &= \alpha_{-J} (e^{B_- x_N} - e^{A_- x_N}) + \beta_{-J} e^{B_- x_N}, \quad \hat{p}_- = \gamma_- e^{Ax_N}. \end{aligned} \quad (3.10)$$

Using the symbols B_{\pm} , we write (3.2) as follows:

$$\begin{cases} \mu_+ B_+^2 \hat{u}_{+j} - \mu_+ D_N^2 \hat{u}_{+j} - \nu_+ i \xi_j (i \xi' \cdot \hat{u}'_+ + D_N \hat{u}_{+N}) = 0 & (x_N > 0), \\ \mu_+ B_+^2 \hat{u}_{+N} - \mu_+ D_N^2 \hat{u}_{+N} - \nu_+ D_N (i \xi' \cdot \hat{u}'_+ + D_N \hat{u}_{+N}) = 0 & (x_N > 0), \\ \mu_- B_-^2 \hat{u}_{-j} - \mu_- D_N^2 \hat{u}_{-j} + i \xi_j \hat{\pi}_- = 0 & (x_N < 0), \\ \mu_- B_-^2 \hat{u}_{-N} - \mu_- D_N^2 \hat{u}_{-N} + D_N \hat{\pi}_- = 0 & (x_N < 0), \\ i \xi' \cdot \hat{u}'_- + D_N \hat{u}_{-N} = 0 & (x_N < 0). \end{cases} \quad (3.11)$$

Substituting the formulas of $\hat{u}_{\pm J}$ into (3.11) and equating the coefficients of $e^{\mp B_{\pm} x_N}$, $e^{-A_+ x_N}$ and $e^{A_+ x_N}$, we have

$$\begin{aligned} \mu_+(A_+^2 - B_+^2)\alpha_{+j} + \nu_+ i\xi_j(i\xi' \cdot \alpha'_+ - A_+\alpha_{+N}) &= 0, \\ \mu_+(A_+^2 - B_+^2)\alpha_{+N} - \nu_+ A_+(i\xi' \cdot \alpha'_+ - A_+\alpha_{+N}) &= 0, \\ i\xi' \cdot \alpha'_+ - \alpha_{+N}B_+ + i\xi' \cdot \beta'_+ - \beta_{+N}B_+ &= 0, \\ \mu_-(A_-^2 - B_-^2)\alpha_{-j} + i\xi_j\gamma_- &= 0, \quad \mu_-(A_-^2 - B_-^2)\alpha_{-N} + A_-\gamma_- = 0, \\ i\xi' \cdot \alpha'_- + \alpha_{-N}B_- + i\xi' \cdot \beta'_- + \beta_{-N}B_- &= 0, \quad i\xi' \cdot \alpha'_- + A_-\alpha_{-N} = 0. \end{aligned} \quad (3.12)$$

First, we represent $i\xi' \cdot \alpha'_{\pm}$, $\alpha_{\pm N}$ and γ_- by $i\xi' \cdot \beta'_{\pm}$ and $\beta_{\pm N}$. It follows from (3.12) that

$$\begin{aligned} i\xi' \cdot \alpha'_+ &= \frac{A_+^2}{A_+B_+ - A_+^2}(i\xi' \cdot \beta'_+ - B_+\beta_{+N}), \quad \alpha_{+N} = \frac{A_+}{A_+B_+ - A_+^2}(i\xi' \cdot \beta'_+ - B_+\beta_{+N}), \\ i\xi' \cdot \alpha'_- &= \frac{A_-}{B_- - A_-}(i\xi' \cdot \beta'_- + B_-\beta_{-N}), \quad \alpha_{-N} = \frac{-1}{B_- - A_-}(i\xi' \cdot \beta'_- + B_-\beta_{-N}), \\ \gamma_- &= -\frac{\mu_-(A_+ + B_-)}{A_-}(i\xi' \cdot \beta'_- + B_-\beta_{-N}). \end{aligned} \quad (3.13)$$

Substituting the relations:

$$\hat{u}_{\pm J}(0) = \beta_{\pm J}, \quad D_N \hat{u}_{+J}(0) = (A_+ - B_+)\alpha_{+J} - B_+\beta_{+J}, \quad D_N \hat{u}_{-J}(0) = (B_- - A_-)\alpha_{-J} + B_-\beta_{-J}$$

into (3.3), we have

$$\begin{aligned} \beta_{+m} &= \beta_{-m} - \hat{h}_m(0), \\ \mu_+((B_+ - A_+)\alpha_{+m} + B_+\beta_{+m} - i\xi_m\beta_{+N}) + \mu_-((B_- - A_-)\alpha_{-m} + B_-\beta_{-m} + i\xi_m\beta_{-N}) &= 0, \\ 2\mu_-((B_- - A_-)\alpha_{-N} + B_-\beta_{-N}) - \gamma_- &= -\sigma_- A_-^2 \hat{H}(0), \\ 2\mu_+((B_+ - A_+)\alpha_{+N} + B_+\beta_{+N}) + (\nu_+ - \mu_+)(-i\xi' \cdot \beta'_+ + (B_+ - A_+)\alpha_{+N} + B_+\beta_{+N}) &= \sigma_+ A_+^2 \hat{H}(0). \end{aligned} \quad (3.14)$$

Using (3.13) and (3.14), we have

$$\begin{aligned} 0 &= L_{11}^+(i\xi' \cdot \beta'_+) + L_{11}^-(i\xi' \cdot \beta'_-) + L_{12}^+\beta_{+N} + L_{12}^-\beta_{-N}, \\ -\sigma_- A_-^3 \hat{H}(0) &= L_{21}^- i\xi' \cdot \beta'_- + L_{22}^- \beta_{-N}, \\ -\sigma_+ A_+^2 \hat{H}(0) &= -L_{21}^+ i\xi' \cdot \beta'_+ - L_{22}^+ \beta_{+N} \end{aligned} \quad (3.15)$$

with

$$\begin{aligned} L_{11}^+ &= \mu_+ \frac{A_+(B_+^2 - A_+^2)}{A_+B_+ - A_+^2}, & L_{11}^- &= \mu_-(A_+ + B_-), \\ L_{12}^+ &= \mu_+ \frac{A_+^2(2A_+B_+ - A_+^2 - B_+^2)}{A_+B_+ - A_+^2}, & L_{12}^- &= \mu_- A_-(B_- - A_-), \\ L_{21}^+ &= 2\mu_+ \frac{A_+(B_+ - A_+)}{A_+B_+ - A_+^2} - (\nu_+ - \mu_+) \frac{A_+^2 - A_-^2}{A_+B_+ - A_+^2}, & L_{21}^- &= \mu_-(B_- - A_-), \\ L_{22}^+ &= (\mu_+ + \nu_+) \frac{B_+(A_+^2 - A_-^2)}{A_+B_+ - A_+^2}, & L_{22}^- &= \mu_-(A_+ + B_-)B_-. \end{aligned} \quad (3.16)$$

As is seen in Sect. 4, we have

$$L_{11}^+, L_{22}^+ \in \mathbf{M}_{1,1}(0), \quad L_{12}^+ \in \mathbf{M}_{2,1}(0), \quad L_{21}^+ \in \mathbf{M}_{0,1}(0), \quad L_{11}^-, L_{21}^- \in \mathbf{M}_{1,2}(0), \quad L_{12}^-, L_{22}^- \in \mathbf{M}_{2,2}(0). \quad (3.17)$$

Using $i\xi' \cdot \beta'_+ = i\xi' \cdot \beta'_- - i\xi' \cdot \hat{h}'(0)$, we write the linear equation (3.15) in the following form:

$$L \begin{pmatrix} i\xi' \cdot \beta'_- \\ \beta_+ \\ \beta_- \end{pmatrix} = \begin{pmatrix} L_{11}^+ i\xi' \cdot \hat{h}'(0) \\ -\sigma_- A_-^3 \hat{H}(0) \\ -\sigma_+ A_+^2 \hat{H}(0) - L_{21}^+ i\xi' \cdot \hat{h}'(0) \end{pmatrix} \quad \text{with} \quad L = \begin{pmatrix} L_{11}^+ + L_{11}^- & L_{12}^+ & L_{12}^- \\ L_{21}^- & 0 & L_{22}^- \\ -L_{21}^+ & -L_{22}^+ & 0 \end{pmatrix}. \quad (3.18)$$

Moreover, we have

$$L^{-1} = \frac{1}{\det L} \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{pmatrix} \quad (3.19)$$

with

$$\begin{aligned} \mathcal{L}_{11} &= L_{22}^+ L_{22}^-, & \mathcal{L}_{12} &= -L_{22}^+ L_{12}^-, & \mathcal{L}_{13} &= L_{12}^+ L_{22}^- \\ \mathcal{L}_{21} &= -L_{21}^+ L_{22}^-, & \mathcal{L}_{22} &= L_{21}^+ L_{12}^-, & \mathcal{L}_{23} &= L_{12}^- L_{21}^- - (L_{11}^+ + L_{11}^-) L_{22}^-, \\ \mathcal{L}_{31} &= -L_{22}^+ L_{21}^-, & \mathcal{L}_{32} &= (L_{11}^+ + L_{11}^-) L_{22}^+ - L_{12}^+ L_{21}^+, & \mathcal{L}_{33} &= -L_{12}^+ L_{21}^-. \end{aligned}$$

By (3.17) and Lemma 3.2, we have

$$\begin{aligned} \mathcal{L}_{11} &\in \mathbf{M}_{3,2}(0), & \mathcal{L}_{12} &\in \mathbf{M}_{3,2}(0), & \mathcal{L}_{13} &\in \mathbf{M}_{4,2}(0) \\ \mathcal{L}_{i1} &\in \mathbf{M}_{2,2}(0), & \mathcal{L}_{i2} &\in \mathbf{M}_{2,2}(0), & \mathcal{L}_{i3} &\in \mathbf{M}_{3,2}(0) \quad (i = 2, 3). \end{aligned} \quad (3.20)$$

The most important fact of this paper is that $\det L \neq 0$ for any $(\lambda, \xi') \in \tilde{\Sigma}_{\epsilon,0}$ and

$$(\det L)^{-1} \in \mathbf{M}_{-4,2}(0). \quad (3.21)$$

From (3.18) and (3.19) it follows that

$$\begin{aligned} i\xi' \cdot \beta'_- &= \frac{A}{\det L} \{(\mathcal{L}_{11} L_{11}^+ - \mathcal{L}_{13} L_{21}^+) i\tilde{\xi}' \cdot \hat{h}'(0) - (\mathcal{L}_{12} \sigma_- A^2 + \mathcal{L}_{13} \sigma_+ A) \hat{H}(0)\}, \\ \beta_{+N} &= \frac{A}{\det L} \{(\mathcal{L}_{21} L_{11}^+ - \mathcal{L}_{23} L_{21}^+) i\tilde{\xi}' \cdot \hat{h}'(0) - (\mathcal{L}_{22} \sigma_- A^2 + \mathcal{L}_{23} \sigma_+ A) \hat{H}(0)\}, \\ \beta_{-N} &= \frac{A}{\det L} \{(\mathcal{L}_{31} L_{11}^+ - \mathcal{L}_{33} L_{21}^+) i\tilde{\xi}' \cdot \hat{h}'(0) - (\mathcal{L}_{32} \sigma_- A^2 + \mathcal{L}_{33} \sigma_+ A) \hat{H}(0)\} \end{aligned} \quad (3.22)$$

where $i\tilde{\xi}' \cdot \hat{k}'(0) = i \sum_{m=1}^{N-1} \xi_m A^{-1} \hat{k}_m(0)$ with $k = g$ and $k = h$. Using the relations: $\beta_{+m} = \beta_{-m} - \hat{h}_m(0)$, by (3.22), we have

$$i\xi' \cdot \beta'_\pm \mp B_\pm \beta_{\pm N} = A \left\{ \sum_{m=1}^{N-1} P_{m,0}^\pm \hat{h}_m(0) + A P_{N,0}^\pm \hat{H}(0) \right\} \quad (3.23)$$

with

$$\begin{aligned} P_{m,0}^+ &= \frac{1}{\det L} \{(\mathcal{L}_{11} - B_+ \mathcal{L}_{21}) L_{11}^+ - (\mathcal{L}_{13} - B_+ \mathcal{L}_{23}) L_{21}^+\} \frac{i\xi_m}{A} - \frac{i\xi_m}{A} \\ P_{N,0}^+ &= \frac{-1}{\det L} \{(\mathcal{L}_{12} - B_+ \mathcal{L}_{22}) \sigma_- A + (\mathcal{L}_{13} - B_+ \mathcal{L}_{23}) \sigma_+\}, \\ P_{m,0}^- &= \frac{1}{\det L} \{(\mathcal{L}_{11} + B_- \mathcal{L}_{31}) L_{11}^+ - (\mathcal{L}_{13} + B_- \mathcal{L}_{33}) L_{21}^+\} \frac{i\xi_m}{A}, \\ P_{N,0}^- &= \frac{-1}{\det L} \{(\mathcal{L}_{12} + B_- \mathcal{L}_{32}) \sigma_- A + (\mathcal{L}_{13} + B_- \mathcal{L}_{33}) \sigma_+\}. \end{aligned}$$

By Lemma 3.2, (3.17), and (3.21), we have

$$P_{m,0}^\pm \in \mathbf{M}_{0,2}(0), \quad P_{N,0}^\pm \in \mathbf{M}_{0,2}(0). \quad (3.24)$$

By (3.13) we have

$$\begin{aligned} \hat{p}_-(x_N) &= -\mu_- \frac{(A + B_-)}{A} (i\xi' \cdot \beta'_- + B_- \beta_{-N}) e^{Ax_N} \\ &= -\mu_- (A + B_-) \left\{ \sum_{m=1}^{N-1} P_{m,0}^- \hat{h}_m(0) + A P_{N,0}^- \hat{H}(0) \right\} e^{Ax_N}, \end{aligned}$$

so that setting $p_{m,1}^- = -\mu_- (A + B_-) P_{m,0}^-$ and $p_{N,1}^- = -\mu_- (A + B_-) P_{N,0}^-$, we have the formula of $\hat{p}_-(x_N)$ in (3.7).

By (3.12), we have

$$\begin{aligned}(B_+ - A_+)\alpha_{+j} &= \frac{\nu_+ i\xi_j}{\mu_+(A_+ + B_+)}(i\xi' \cdot \alpha'_+ - A_+\alpha_{+N}), \\(B_+ - A_+)\alpha_{+N} &= -\frac{\nu_+ A_+}{\mu_+(A_+ + B_+)}(i\xi' \cdot \alpha'_+ - A_+\alpha_{+N}), \\(B_- - A_-)\alpha_{-j} &= -\frac{i\xi_j}{A}(i\xi' \cdot \beta'_- + B_-\beta_{-N}), \quad (B_- - A_-)\alpha_{-N} = -(i\xi' \cdot \beta'_- + B_-\beta_{-N}).\end{aligned}$$

Since $i\xi' \cdot \alpha'_+ - A_+\alpha_{+N} = \frac{A^2 - A_+^2}{A_+ B_+ - A^2}(i\xi' \cdot \beta'_+ - B_+\beta_{-N})$ as follows from (3.13), setting $A_- = A$, by (3.23) we have

$$(B_{\pm} - A_{\pm})\alpha_{\pm J} = A\left\{\sum_{m=1}^{N-1} R_{Jm,0}^{\pm} \hat{h}_m(0) + AR_{JN,0}^{\pm} \hat{H}(0)\right\} \quad (3.25)$$

with

$$\begin{aligned}R_{jm,0}^+ &= \frac{\nu_+ i\xi_j P_{m,0}^+}{\mu_+(A_+ + B_+)} \frac{A^2 - A_+^2}{A_+ B_+ - A^2}, \quad R_{jN,0}^+ = \frac{\nu_+ i\xi_j P_{N,0}^+}{\mu_+(A_+ + B_+)} \frac{A^2 - A_+^2}{A_+ B_+ - A^2}, \\R_{Nm,0}^+ &= -\frac{\nu_+ A_+ P_{m,0}^+}{\mu_+(A_+ + B_+)} \frac{A^2 - A_+^2}{A_+ B_+ - A^2}, \quad R_{NN,0}^+ = -\frac{\nu_+ A_+ P_{N,0}^+}{\mu_+(A_+ + B_+)} \frac{A^2 - A_+^2}{A_+ B_+ - A^2}, \\R_{jm,0}^- &= -\frac{i\xi_j}{A} P_{m,0}^+, \quad R_{jN,1}^- = -\frac{i\xi_j}{A} P_{N,1}^-, \quad R_{Nm,0}^- = -P_{m,0}^-, \quad R_{NN,0}^- = -P_{N,0}^-.\end{aligned}$$

Recalling $A_- = A$, we have $(e^{\mp B_{\pm} x_N} - e^{\mp A_{\pm} x_N})\alpha_{\pm J} = M_{\pm}(x_N)(B_{\pm} - A_{\pm})\alpha_{\pm J}$. Thus, if we set $\hat{u}_{J1}^{\pm} = AM_{\pm}(x_N)(\sum_{m=1}^{N-1} R_{Jm,0}^{\pm} \hat{h}_m(0) + R_{JN,1}^{\pm} \hat{H}(0))$, then $\hat{u}_{J1}^{\pm} = \alpha_{\pm J}(e^{\mp B_{\pm} x_N} - e^{\mp A_{\pm} x_N})$. As is seen in Sect. 4 below, we have

$$A_+ \in \mathbf{M}_{1,1}(0), \quad B_+ \in \mathbf{M}_{1,1}(0), \quad (A_+ + B_+)^{-1} \in \mathbf{M}_{-1,1}(0), \quad \frac{A^2 - A_+^2}{A_+ B_+ - A^2} \in \mathbf{M}_{0,1}(0), \quad (3.26)$$

which, combined with (3.24), furnishes that $R_{jm,0}^+ \in \mathbf{M}_{0,2}(0)$ and $R_{jN,1}^+ \in \mathbf{M}_{1,2}(0)$. And also, by (3.24) and (3.26), we have $R_{jm,0}^- \in \mathbf{M}_{0,2}(0)$ and $R_{jN,1}^- \in \mathbf{M}_{1,2}(0)$.

From (3.14) it follows that

$$\begin{aligned}\beta_{\pm j} &= \frac{\mp \mu_{\mp} B_{\mp}}{\mu_+ B_+ + \mu_- B_-} \hat{h}_j(0) \\&+ \frac{1}{\mu_+ B_+ + \mu_- B_-} \{-\mu_+(B_+ - A)\alpha_{+j} - \mu_-(B_- - A)\alpha_{-j} + i\xi_j \mu_+ \beta_{+N} - i\xi_j \mu_- \beta_{-N}\}.\end{aligned}$$

We set

$$T_{j,0}^{\pm} = \frac{\mp \mu_{\mp} B_{\mp}}{\mu_+ B_+ + \mu_- B_-}$$

and in view of (3.25) and (3.22) we set

$$\begin{aligned}S_{jm,-1}^{\pm} &= \frac{-1}{\mu_+ B_+ + \mu_- B_-} \left(\mu_+ R_{jm,0}^+ + \mu_- R_{jm,0}^- \right. \\&\quad \left. - \frac{\mu_+ i\xi_j (\mathcal{L}_{21} L_{11}^+ - \mathcal{L}_{23} L_{21}^+)}{\det L} \frac{i\xi_m}{A} + \frac{\mu_- i\xi_j (\mathcal{L}_{31} L_{11}^+ - \mathcal{L}_{33} L_{21}^+)}{\det L} \frac{i\xi_m}{A} \right), \\S_{jN,-1}^{\pm} &= \frac{-1}{\mu_+ B_+ + \mu_- B_-} \left(\mu_+ R_{jN,0}^+ + \mu_- R_{jN,0}^- \right. \\&\quad \left. + \frac{\mu_+ i\xi_j (\mathcal{L}_{22} \sigma_- A + \mathcal{L}_{23} \sigma_+)}{\det L} - \frac{\mu_- i\xi_j (\mathcal{L}_{32} \sigma_- A + \mathcal{L}_{33} \sigma_+)}{\det L} \right).\end{aligned}$$

Thus, if we set $\hat{u}_{j2}^{\pm} = Ae^{\mp B_{\pm} x_N} \{\sum_{m=1}^{N-1} S_{jm,-1}^{\pm} \hat{h}_m(0) + AS_{jN,-1}^{\pm} \hat{H}(0)\}$ and $\hat{u}_{j3}^{\pm} = e^{\mp B_{\pm} x_N} T_{j,0}^{\pm} \hat{h}_j(0)$, then we have $\beta_{\pm j} e^{\mp B_{\pm} x_N} = \hat{u}_{j2}^{\pm} + \hat{u}_{j3}^{\pm}$. Moreover, by (3.17), (3.20), (3.21), (3.24), (3.26), we have $S_{jm,-1}^{\pm} \in \mathbf{M}_{-1,2}(0)$, $S_{jN,-1}^{\pm} \in \mathbf{M}_{-1,2}(0)$, and $T_{j,0}^{\pm} \in \mathbf{M}_{0,1}(0)$.

Finally, in view of (3.22), we define $\hat{u}_{N2}^\pm = Ae^{\mp B_\pm x_N} \{ \sum_{m=1}^{N-1} S_{Nm,-1}^\pm \hat{h}_m(0) + AS_{NN,-1}^\pm \hat{H}(0) \}$ with

$$\begin{aligned} S_{Nm,-1}^+ &= \frac{\mathcal{L}_{21}L_{11}^+ - \mathcal{L}_{23}L_{21}^+}{\det L} \frac{i\xi_m}{A}, & S_{NN,-1}^+ &= \frac{-(\mathcal{L}_{22}\sigma_- A + \mathcal{L}_{23}\sigma_+)}{\det L}, \\ S_{Nm,-1}^- &= \frac{\mathcal{L}_{31}L_{11}^+ - \mathcal{L}_{33}L_{21}^+}{\det L} \frac{i\xi_m}{A}, & S_{NN,-1}^- &= \frac{-(\mathcal{L}_{32}\sigma_- A + \mathcal{L}_{33}\sigma_+)}{\det L}. \end{aligned}$$

Thus, we have $\hat{u}_{N2}^\pm = \beta_{\pm N} e^{\mp B_\pm x_N}$. Moreover, by (3.17), (3.21) and (3.20), we have $S_{Nm,-1}^\pm \in \mathbf{M}_{-1,2}(0)$ and $S_{NN,-1}^\pm \in \mathbf{M}_{-1,1}(0)$. Summing up, we have obtained (3.7) and (3.8).

To prove Theorem 2.8, we consider problem (2.4), namely problem (3.1) with $H = 0$. To construct our solution operator from the solution formulas in (3.7) with $\hat{H} = 0$, first of all we observe the following formulas due to Volevich:

$$a(\xi', x_N) \hat{h}(0) = - \int_0^{\pm\infty} \{ (\partial_N a)(\xi', x_N + y_N) \hat{h}(y_N) + a(\xi', x_N + y_N) \widehat{\partial_N h}(\xi', y_N) \} dy_N,$$

where $\partial_j = \partial/\partial x_j$. Using the identity: $1 = \frac{\rho_\pm \lambda}{\mu_\pm B_\pm^2} - \sum_{k=1}^{N-1} \frac{(i\xi_k)(i\xi_k)}{B_\pm^2}$, we write

$$\begin{aligned} a(\xi', x_N) \hat{h}(\xi', 0) &= - \int_0^{\pm\infty} a(\xi', x_N + y_N) \left[\frac{\rho_\pm \lambda^{1/2} \widehat{\partial_N h}(\xi', y_N)}{\mu_\pm B_\pm^2} - \sum_{k=1}^{N-1} \frac{i\xi_k \widehat{\partial_k \partial_N h}(\xi', y_N)}{B_\pm} \right] dy_N \\ &\quad - \int_0^{\pm\infty} (\partial_N a)(\xi', x_N + y_N) \left[\frac{\rho_\pm \lambda \hat{h}(\xi', y_N)}{\mu_\pm B_\pm^2} - \sum_{k=1}^{N-1} \frac{\widehat{\partial_k \partial_k h}(\xi', y_N)}{B_\pm^2} \right] dy_N. \end{aligned}$$

Let $f_3, f_4 = (f_{41}, \dots, f_{4N})$ and $f_5 = (f_{5JK} \mid J, K = 1, \dots, N)$ be the corresponding variables to λh , $\lambda^{1/2} \nabla h$ and $\nabla^2 h = (\partial_J \partial_K h \mid J, K = 1, \dots, N)$. If we define $A_1^\pm[a](f_3, f_4, f_5)$ by

$$\begin{aligned} A_1^\pm[a](f_3, f_4, f_5) &= - \int_0^{\pm\infty} \mathcal{F}_{\xi'}^{-1} \left[a(\xi', x_N + y_N) \left\{ \frac{\rho_\pm \hat{f}_{4N}(\xi', y_N)}{\mu_\pm B_\pm^2} - \sum_{k=1}^{N-1} \frac{i\xi_k \hat{f}_{3kN}(\xi', y_N)}{B_\pm} \right\} \right] dy_N \\ &\quad - \int_0^{\pm\infty} \mathcal{F}_{\xi'}^{-1} \left[(\partial_N a)(\xi', x_N + y_N) \left\{ \frac{\rho_\pm \hat{f}_3(\xi', y_N)}{\mu_\pm B_\pm^2} - \sum_{k=1}^{N-1} \frac{\hat{f}_{3kk}(\xi', y_N)}{B_\pm^2} \right\} \right] dy_N, \end{aligned} \quad (3.27)$$

then we have

$$\mathcal{F}_{\xi'}^{-1} [a(\xi', x_N) \hat{h}(\xi', 0)] = A_1^\pm[a](\lambda h, \lambda^{1/2} \nabla h, \nabla^2 h). \quad (3.28)$$

Let us define u_{ji}^\pm ($i = 1, 2, 3$) and p_- by $u_{ji}^\pm = \mathcal{F}_{\xi'}^{-1} [\hat{u}_{ji}^\pm]$ ($i = 1, 2, 3$) and $p_- = \mathcal{F}_{\xi'}^{-1} [\hat{p}_-]$ with $\hat{H} = 0$. Setting $u_{J\pm} = \sum_{i=1}^3 u_{ji}^\pm$, by (3.7) we see that $\mathbf{u}_\pm = (u_{1\pm}, \dots, u_{N\pm})$ and p_- satisfy the equations (2.4). According to the formulas (3.27) and (3.28), we define our solution operators $\mathcal{S}_{ji}^\pm(\lambda)$ ($i = 1, 2, 3$) and $\mathcal{P}_{-2}(\lambda)$ of problem (2.4) such that

$$\begin{aligned} u_{ji}^\pm &= \mathcal{S}_{ji}^\pm(\lambda)(\lambda \mathbf{h}, \lambda^{1/2} \nabla \mathbf{h}, \nabla^2 \mathbf{h}) \quad \text{on } \mathbb{R}_\pm^N \quad (i = 1, 2, 3), \\ p_- &= \mathcal{P}_{-2}(\lambda)(\lambda \mathbf{h}, \lambda^{1/2} \nabla \mathbf{h}, \nabla^2 \mathbf{h}) \quad \text{on } \mathbb{R}_-^N \end{aligned} \quad (3.29)$$

with $\mathbf{h} = (h_1, \dots, h_{N-1})$ as follows: Note that

$$\begin{aligned} \partial_N M_\pm(x_N + y_N) &= \mp(e^{\pm B_\pm(x_N + y_N)} + A_\pm M_\pm(x_N + y_N)), & \partial_N e^{A(x_N + y_N)} &= A e^{A(x_N + y_N)}, \\ \partial_N e^{\mp B_\pm(x_N + y_N)} &= \mp B_\pm e^{\mp B_\pm(x_N + y_N)}, \end{aligned} \quad (3.30)$$

where we have set $A_- = A$. Let $F_3 = (F_{3m} \mid m = 1, \dots, N-1)$, $F_4 = (F_{4Jm} \mid J = 1, \dots, N, m = 1, \dots, N-1)$ and $F_5 = (F_{5JKm} \mid J, K = 1, \dots, N, m = 1, \dots, N-1)$ be the corresponding variables to $\lambda \mathbf{h} = (\lambda h_1, \dots, \lambda h_{N-1})$, $\lambda^{1/2} \nabla \mathbf{h} = (\lambda^{1/2} \partial_J h_m \mid J = 1, \dots, N, m = 1, \dots, N-1)$ and $\nabla^2 \mathbf{h} = (\partial_J \partial_K h_m \mid J, K = 1, \dots, N, m = 1, \dots, N-1)$, respectively. Then, we define the operators $\mathcal{S}_{j1}^\pm(\lambda)$, $\mathcal{S}_{j2}^\pm(\lambda)$, $\mathcal{S}_{j3}^\pm(\lambda)$, and $\mathcal{P}_{-2}(\lambda)$ by

$$\mathcal{S}_{j1}^\pm(\lambda)(F_3, F_4, F_5) =$$

$$\begin{aligned}
& - \int_0^{\pm\infty} \mathcal{F}_{\xi'}^{-1} \left[AM_{\pm}(x_N + y_N) \sum_{m=1}^{N-1} \left(\frac{R_{Jm,0}^{\pm} \rho_{\pm} \lambda^{1/2}}{\mu_{\pm} B_{\pm}^2} \hat{F}_{4Nm}(\xi', y_N) - \sum_{k=1}^{N-1} \frac{R_{Jm,0}^{\pm} (i\xi_k)}{B_{\pm}^2} \hat{F}_{5kNm}(\xi', y_N) \right) \right. \\
& \quad \mp AM_{\pm}(x_N + y_N) \sum_{m=1}^{N-1} \left(\frac{A_{\pm} R_{Jm,0}^{\pm} \rho_{\pm}}{\mu_{\pm} B_{\pm}^2} \hat{F}_{3m}(\xi', y_N) - \sum_{k=1}^{N-1} \frac{A_{\pm} R_{Jm,0}^{\pm}}{B_{\pm}^2} \hat{F}_{5kkm}(\xi', y_N) \right) \\
& \quad \mp Ae^{\mp B_{\pm} x_N} \sum_{m=1}^{N-1} \left(\frac{R_{Jm,0}^{\pm} \rho_{\pm}}{\mu_{\pm} B_{\pm}^2} \hat{F}_{3m}(\xi', y_N) - \sum_{k=1}^{N-1} \frac{R_{Jm,0}^{\pm}}{B_{\pm}^2} \hat{F}_{5kkm}(\xi', y_N) \right) \left. \right] (x') dy_N; \\
\mathcal{S}_{J2}^{\pm}(\lambda)(F_3, F_4, F_5) &= \\
& - \int_0^{\pm\infty} \mathcal{F}_{\xi'}^{-1} \left[Ae^{\mp B_{\pm}(x_N + y_N)} \sum_{m=1}^{N-1} \left(\frac{S_{Jm,-1}^{\pm} \rho_{\pm} \lambda^{1/2}}{\mu_{\pm} B_{\pm}^2} \hat{F}_{4Nm}(\xi', y_N) - \sum_{k=1}^{N-1} \frac{S_{Jm,-1}^{\pm} i\xi_k}{B_{\pm}^2} \hat{F}_{5kNm}(\xi', y_N) \right) \right. \\
& \quad \mp Ae^{\mp B_{\pm}(x_N + y_N)} \sum_{m=1}^{N-1} \left(\frac{S_{Jm,-1}^{\pm} \rho_{\pm}}{\mu_{\pm} B_{\pm}^2} \hat{F}_{3m}(\xi', y_N) - \sum_{k=1}^{N-1} \frac{S_{Jm,-1}^{\pm}}{B_{\pm}^2} \hat{F}_{5kkm}(\xi', y_N) \right) \left. \right] (x') dy_N; \\
\mathcal{S}_{J5}^{\pm}(\lambda)(F_3, F_4, F_5) &= \\
& - \int_0^{\pm\infty} \mathcal{F}_{\xi'}^{-1} \left[e^{\mp B_{\pm}(x_N + y_N)} \left\{ \frac{T_{j,0}^{\pm} \rho_{\pm}}{\mu_{\pm} B_{\pm}^2} \hat{F}_{4Nj}(\xi', y_N) - \sum_{k=1}^{N-1} \frac{T_{j,0}^{\pm} i\xi_k}{B_{\pm}^2} \hat{F}_{5kNj}(\xi', y_N) \right. \right. \\
& \quad \left. \left. \pm \left(\frac{T_{j,0}^{\pm} \rho_{\pm}}{\mu_{\pm} B_{\pm}^2} \hat{F}_{3j}(\xi', y_N) - \sum_{k=1}^{N-1} \frac{T_{j,0}^{\pm}}{B_{\pm}^2} \hat{F}_{5kkj}(\xi', y_N) \right) \right\} \right] (x') dy_N; \\
\mathcal{S}_{N3}^{\pm}(\lambda)(F_3, F_4, F_5) &= 0 \\
\mathcal{P}_{-2}(\lambda)(F_3, F_4, F_5) &= \\
& \int_{-\infty}^0 \left[e^{A(x_N + y_N)} \left\{ \sum_{m=1}^{N-1} \left(\frac{p_{m,1}^{-} \rho_{-} \lambda^{1/2}}{\mu_{-} B_{-}^2} \hat{F}_{4Nm} - \sum_{k=1}^{N-1} \frac{p_{m,1}^{-} i\xi_k}{B_{-}^2} \hat{F}_{5kNm}(\xi', y_N) \right) \right. \right. \\
& \quad \left. \left. + \sum_{m=1}^{N-1} \left(\frac{Ap_{m,1}^{-} \rho_{-}}{\mu_{-} B_{-}^2} \hat{F}_{3m}(\xi', y_N) - \sum_{k=1}^{N-1} \frac{Ap_{m,1}^{-}}{B_{-}^2} \hat{F}_{5kkm}(\xi', y_N) \right) \right\} \right] (x') dy_N. \quad (3.31)
\end{aligned}$$

Obviously, by (3.28), we have (3.29).

Given that operators $\mathcal{A}_{\pm 2}(\lambda)$ are defined by $\mathcal{A}_{\pm 2}(\lambda)\mathbf{F}' = \sum_{i=1}^3 (\mathcal{S}_{1i}^{\pm}(\lambda)\mathbf{F}', \dots, \mathcal{S}_{N_i}^{\pm}(\lambda)\mathbf{F}')$ with $\mathbf{F}' = (F_3, F_4, F_5)$, by (3.29) we have

$$\mathbf{u}_{\pm} = \mathcal{A}_{\pm 2}(\lambda)(\lambda \mathbf{h}, \lambda^{1/2} \nabla \mathbf{h}, \nabla^2 \mathbf{h}), \quad p_{-} = \mathcal{P}_{-2}(\lambda)(\lambda \mathbf{h}, \lambda^{1/2} \nabla \mathbf{h}, \nabla^2 \mathbf{h}). \quad (3.32)$$

Moreover, we have Theorem 2.8 with the help of the following two lemmas:

Lemma 3.4. *Let $1 < q < \infty$ and let n_1^{+} , n_2^{+} and n_3^{+} be multipliers belonging to $\mathbf{M}_{-1,2}(0)$, $\mathbf{M}_{-2,2}(0)$ and $\mathbf{M}_{-1,1}(0)$, respectively. Let K_i^{+} ($i = 1, 2, 3$) be operators defined by*

$$\begin{aligned}
K_1^{+}(\lambda)g &= \int_0^{\infty} \mathcal{F}_{\xi'}^{-1} [n_1^{+}(\lambda, \xi') AM_{+}(x_N + y_N) \hat{g}(\xi', y_N)](x') dy_N, \\
K_2^{+}(\lambda)g &= \int_0^{\infty} \mathcal{F}_{\xi'}^{-1} [n_2^{+}(\lambda, \xi') Ae^{-B_{+}(x_N + y_N)} \hat{g}(\xi', y_N)](x') dy_N, \\
K_3^{+}(\lambda)g &= \int_0^{\infty} \mathcal{F}_{\xi'}^{-1} [n_3^{+}(\lambda, \xi') e^{-B_{+}(x_N + y_N)} \hat{g}(\xi', y_N)](x') dy_N.
\end{aligned}$$

Then, there exists a constant C such that

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_+^N), L_q(\mathbb{R}_+^N)^{1+N+N^2})}(\{(\tau \partial_{\tau})^{\ell} G_{\lambda}^1 K_i^{+}(\lambda) \mid \lambda \in \Lambda\}) \leq C \quad (\ell = 0, 1, \quad i = 1, 2, 3).$$

Lemma 3.5. Let $1 < q < \infty$ and let n_1^-, n_2^-, n_3^- and n_4^- be multipliers belonging to $\mathbf{M}_{-1,2}(0)$, $\mathbf{M}_{-2,2}(0)$, $\mathbf{M}_{-1,1}(0)$ and $\mathbf{M}_{0,2}(0)$, respectively. Let K_i^+ ($i = 1, 2, 3, 4$) be operators defined by

$$\begin{aligned} K_1^-(\lambda)g &= \int_{-\infty}^0 \mathcal{F}_{\xi'}^{-1}[n_1^-(\lambda, \xi')AM_-(x_N + y_N)\hat{g}(\xi', y_N)](x') dy_N, \\ K_2^-(\lambda)g &= \int_{-\infty}^0 \mathcal{F}_{\xi'}^{-1}[n_2^-(\lambda, \xi')Ae^{B_-(x_N + y_N)}\hat{g}(\xi', y_N)](x') dy_N, \\ K_3^-(\lambda)g &= \int_{-\infty}^0 \mathcal{F}_{\xi'}^{-1}[n_3^-(\lambda, \xi')e^{B_-(x_N + y_N)}\hat{g}(\xi', y_N)](x') dy_N, \\ K_4^-(\lambda)g &= \int_{-\infty}^0 \mathcal{F}_{\xi'}^{-1}[n_4^-(\lambda, \xi')e^{A(x_N + y_N)}\hat{g}(\xi', y_N)](x') dy_N. \end{aligned}$$

Then, there exists a constant C such that

$$\begin{aligned} \mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_-^N), L_q(\mathbb{R}_-^N)^{1+N+N^2})}(\{(\tau\partial_\tau)^\ell G_\lambda K_i^-(\lambda) \mid \lambda \in \Lambda\}) &\leq C \quad (\ell = 0, 1, \quad i = 1, 2, 3), \\ \mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_-^N), L_q(\mathbb{R}_-^N)^N)}(\{(\tau\partial_\tau)^\ell \nabla K_4^-(\lambda) \mid \lambda \in \Lambda\}) &\leq C \quad (\ell = 0, 1). \end{aligned} \quad (3.33)$$

Remark 3.6. Lemma 3.4 was proved in [4, Sect.2] and Lemme 3.5 was proved in [14].

4 Some estimates of several multipliers

In this section, we prove (3.17) and (3.26). For this purpose, we use the following well-known estimate:

$$|\alpha\lambda + \beta| \geq (\sin \frac{\epsilon}{2})(\alpha|\lambda| + \beta) \quad (4.1)$$

for any $\lambda \in \Sigma_\epsilon$ and positive numbers α and β .

First we estimate A_+^s , B_\pm^s , $(A_+ + B_+)^s$ and $(\mu_+ B_+ + \mu_- B_-)^s$. For this purpose, we use the estimates:

$$c(|\lambda|^{1/2} + A) \leq \operatorname{Re} M_1 \leq |M_1| \leq c'(|\lambda|^{1/2} + A) \quad (M_1 = A_+, \quad B_\pm) \quad (4.2)$$

for any $(\lambda, \xi') \in \tilde{\Sigma}_\epsilon = \Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\})$ with some positive constants c and c' , which immediately follows from (4.1). Here and in the sequel, c and c' denote some positive constants essentially depending on μ_\pm , ν_+ , ρ_\pm and ϵ . In particular, by (4.2) we have

$$c(|\lambda|^{1/2} + A) \leq \operatorname{Re} M_2 \leq |M_2| \leq c'(|\lambda|^{1/2} + A) \quad (M_2 = A_+ + B_+, \quad \mu_+ B_+ + \mu_- B_-) \quad (4.3)$$

for any $(\lambda, \xi') \in \tilde{\Sigma}_{\epsilon,0}$. As was seen in Enomoto and Shibata [2, Lemma 4.3], using (4.2), (4.3) and the Bell formula:

$$\partial_{\xi'}^{\kappa'} f(g(\xi')) = \sum_{\ell=1}^{|\kappa'|} f^{(\ell)}(g(\xi')) \sum_{\substack{\kappa'_1 + \dots + \kappa'_\ell = \kappa' \\ |\kappa'_i| \geq 1}} \Gamma_{\kappa'_1, \dots, \kappa'_\ell}^{\kappa'} (\partial_{\xi'}^{\kappa'_1} g(\xi')) \cdots (\partial_{\xi'}^{\kappa'_\ell} g(\xi')) \quad (4.4)$$

with suitable coefficients $\Gamma_{\kappa'_1, \dots, \kappa'_\ell}^{\kappa'}$, where $f^{(\ell)}(t) = d^\ell f(t)/dt^\ell$, we see that

$$(M_3)^s \in \mathbf{M}_{s,1}(0) \quad (M_3 = A_+, \quad B_+, \quad A_+ + B_+, \quad \mu_+ B_+ + \mu_- B_-). \quad (4.5)$$

Second, we estimate $(A_+ B_+ - A^2)^{-1}$. For this purpose, we write

$$\frac{1}{A_+ B_+ - A^2} = \frac{(\mu_+ + \nu_+)\mu_+}{\rho_+(2\mu_+ + \nu_+)\lambda} P(\lambda, \xi') \quad \text{with } P(\lambda, \xi') = \frac{A_+ B_+ + A^2}{\rho_+(2\mu_+ + \nu_+)^{-1}\lambda + A^2}. \quad (4.6)$$

Noting that $A^2 \in \mathbf{M}_{2,1}(0)$, by (4.1), (4.4) and (4.5) we have

$$A_+ B_+ + A^2 \in \mathbf{M}_{2,1}(0), \quad (\rho_+(2\mu_+ + \nu_+)^{-1}\lambda + A^2)^s \in \mathbf{M}_{2s,1}(0), \quad (4.7)$$

so that by Lemma 3.2 we have

$$P \in \mathbf{M}_{0,1}(0). \quad (4.8)$$

Since $A^2 - A_+^2 = \rho_+(\mu_+ + \nu_+)^{-1}\lambda$, by (4.6) and (4.8), we have $\frac{A^2 - A_+^2}{A_+ B_+ - A^2} \in \mathbf{M}_{0,1}(0)$, which, combined with (4.5), furnishes (3.26).

Applying (4.6) to the formula in (3.16), we have

$$\begin{aligned} L_{11}^+ &= \frac{\mu_+(\mu_+ + \nu_+)}{2\mu_+ + \nu_+} A_+ P, & L_{12}^+ &= \mu_+ A^2 \left(2 - \frac{\mu_+ + \nu_+}{2\mu_+ + \nu_+} P \right), \\ L_{21}^+ &= \left(\frac{2\mu_+ \nu_+}{2\mu_+ + \nu_+} \frac{A_+}{B_+ + A_+} - \frac{\mu_+(\nu_+ - \mu_+)}{2\mu_+ + \nu_+} \right) P, & L_{22}^+ &= \frac{\mu_+(\mu_+ + \nu_+)}{2\mu_+ + \nu_+} B_+ P. \end{aligned} \quad (4.9)$$

By Lemma 3.2, (3.16), (4.5), (4.8) and (4.9), we have $L_{11}^+ \in \mathbf{M}_{1,1}(0)$, $L_{12}^+ \in \mathbf{M}_{2,1}(0)$, $L_{21}^+ \in \mathbf{M}_{0,1}(0)$ and $L_{22}^+ \in \mathbf{M}_{1,1}(0)$. In addition, since $A \in \mathbf{M}_{1,2}(0)$ and $B_- \in \mathbf{M}_{1,2}(0)$, by Lemma 3.2 we have $A \pm B_- \in \mathbf{M}_{1,2}(0)$ and $(A + B_-)B_- \in \mathbf{M}_{2,2}(0)$. Summing up, we have proved (3.17).

5 Analysis of Lopatinski determinant

In this section, we show the following lemma which implies (3.21).

Lemma 5.1. *Let L be the matrix defined in (3.18). Then, there exists a positive constant ω depending solely on μ_\pm , ν_\pm , ρ_\pm , and ϵ such that*

$$|\det L| \geq \omega(|\lambda|^{1/2} + A)^4 \quad (5.1)$$

for any $(\lambda, \xi') \in \tilde{\Sigma}_{\epsilon,0}$.

Moreover, we have

$$|\partial_{\xi'}^{\kappa'} \{(\tau \partial_\tau)^\ell (\det L)^{-1}\}| \leq C_{\kappa'} (|\lambda|^{1/2} + A)^{-4} A^{-|\kappa'|} \quad (\ell = 0, 1) \quad (5.2)$$

for any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$ and $(\lambda, \xi') \in \tilde{\Sigma}_{\epsilon,0}$. Namely, $(\det L)^{-1} \in \mathbf{M}_{-4,2}(0)$.

Proof. We see that

$$\det L = L_{22}^- \det L^+ + L_{22}^+ \det L^- \quad (5.3)$$

with $L^\pm = \det \begin{pmatrix} L_{11}^\pm & L_{12}^\pm \\ L_{21}^\pm & L_{22}^\pm \end{pmatrix}$. To prove (5.1), first we consider the case: $R_1 |\lambda|^{1/2} \leq A$ with large $R_1 \geq 1$. Let P be the function defined in (4.6). By (4.6) we see easily that $P = 2 + O(\delta_1)$, that $A_+ = A(1 + O(\delta_1))$ and that $B_\pm = A(1 + O(\delta_1))$ when $|\rho_+(\mu_+ + \nu_+)^{-1}\lambda A^{-2}| \leq \rho_+(\mu_+ + \nu_+)^{-1} R_1^{-2} \leq \delta_1^2$ and $|\rho_\pm \mu_\pm^{-1} \lambda A^{-2}| \leq \rho_\pm \mu_\pm^{-1} R_1^{-2} \leq \delta_1^2$ with very small positive number δ_1 . Thus, by (4.9) we have

$$\begin{aligned} L_{11}^+ &= \frac{2\mu_+(\mu_+ + \nu_+)}{2\mu_+ + \nu_+} A(1 + O(\delta_1)), & L_{12}^+ &= \frac{2(\mu_+)^2}{2\mu_+ + \nu_+} A^2(1 + O(\delta_1)), \\ L_{21}^+ &= \frac{2(\mu_+)^2}{2\mu_+ + \nu_+} (1 + O(\delta_1)), & L_{22}^+ &= \frac{2\mu_+(\mu_+ + \nu_+)}{2\mu_+ + \nu_+} A(1 + O(\delta_1)). \end{aligned} \quad (5.4)$$

On the other hand, we have $B_- - A = \frac{\gamma_0 - \lambda}{\mu_0 - (B_- + A)} = AO(\delta_1)$, so that by (3.16) we have

$$\begin{aligned} L_{11}^- &= 2\mu_- A(1 + O(\delta_1)), & L_{12}^- &= A^2 O(\delta_1), \\ L_{21}^- &= AO(\delta_1), & L_{22}^- &= 2\mu_- A^2(1 + O(\delta_1)). \end{aligned} \quad (5.5)$$

Thus, by (5.3) we have

$$\det L = \omega_1 A^4 (1 + O(\delta_1)) \quad \text{with } \omega_1 = \frac{8\mu_+ \mu_- (\mu_+ \nu_+ + \mu_- (\mu_+ + \nu_+))}{2\mu_+ + \nu_+} \quad (5.6)$$

so that we can choose $R_1 \geq 1$ so large that

$$|\det L| \geq \frac{1}{2}\omega_1(|\lambda|^{1/2} + A)^4 \quad (5.7)$$

for any $(\lambda, \xi') \in \tilde{\Sigma}_{\epsilon,0}$ with $R_1|\lambda|^{1/2} \leq A$.

Secondly, we consider the case: $R_2 A \leq |\lambda|^{1/2}$ with large $R_2 \geq 1$. In this case, we have

$$A_+ = (\mu_+ + \nu_+ + \delta)^{-1/2}(\gamma_{0+}\lambda)^{1/2}(1 + O(\delta_2)), \quad B_{\pm} = (\mu_{\pm})^{-1/2}(\gamma_{0\pm}\lambda)^{1/2}(1 + O(\delta_2)) \quad (5.8)$$

when $|(\mu_+ + \nu_+)(\rho_+\lambda)^{-1}A^2| \leq (\mu_+ + \nu_+ + \delta_1)\rho_+^{-1}R_2^{-2} \leq \delta_2^2$ and $|\mu_{\pm}(\rho_{\pm}\lambda)^{-1}A^2| \leq \mu\rho_{\pm}^{-1}R_2^{-2} \leq \delta_2^2$ with some very small positive number δ_2 . By (3.16)

$$\begin{aligned} L_{11}^+ &= (\mu_+\rho_+\lambda)^{1/2}(1 + O(\delta_2)) & L_{12}^+ &= O(\delta_2)\lambda \\ L_{21}^+ &= \frac{2\mu_+((\mu_+ + \nu_+)^{1/2} - \mu_+^{1/2}) - (\nu_+ - \mu_+)\mu_+^{1/2}}{(\mu_+ + \nu_+)^{1/2}}(1 + O(\delta_2)), & L_{22}^+ &= ((\mu_+ + \nu_+)\rho_+\lambda)^{1/2}(1 + O(\delta_2)) \\ L_{11}^- &= (\mu_-\rho_-\lambda)^{1/2}(1 + O(\delta_2)) & L_{12}^- &= O(\delta_2)\lambda \\ L_{21}^- &= (\mu_-\rho_-\lambda)^{1/2}(1 + O(\delta_2)), & L_{22}^- &= \rho_+\lambda(1 + O(\delta_2)) \end{aligned} \quad (5.9)$$

Thus, by (5.3) we have

$$|\det L| = \omega_2|\lambda|^2(1 + O(\delta_2)) \quad \text{with } \omega_2 = \mu_+^{1/2}(\mu_+ + \nu_+)^{1/2}\rho_+\rho_- + \mu_-^{1/2}(\mu_+ + \nu_+)^{1/2}\rho_+^{1/2}\rho_-^{3/2}, \quad (5.10)$$

so that we can choose $R_2 \geq 1$ so large that

$$|\det L| \geq \frac{1}{2}\omega_2(|\lambda|^{1/2} + A)^4 \quad (5.11)$$

for any $(\lambda, \xi') \in \tilde{\Sigma}_{\epsilon,0}$ with $R_2 A \leq |\lambda|^{1/2}$.

Thirdly, we consider the case: $R_2^{-1}|\lambda|^{1/2} \leq A \leq R_1|\lambda|^{1/2}$. Set

$$\begin{aligned} \tilde{\lambda} &= \frac{\lambda}{(|\lambda|^{1/2} + A)^2}, \quad \tilde{A} = \frac{A}{|\lambda|^{1/2} + A}, \\ \tilde{A}_+ &= \sqrt{\rho_+(\mu_+ + \nu_+)^{-1}\tilde{\lambda} + \tilde{A}^2}, \quad \tilde{B}_{\pm} = \sqrt{\rho_{\pm}(\mu_{\pm})^{-1}\tilde{\lambda} + \tilde{A}^2}, \\ D_{\epsilon}(R_1, R_2) &= \{(\tilde{\lambda}, \tilde{A}) \mid (1 + R_1)^{-2} \leq |\tilde{\lambda}| \leq R_2^2(1 + R_2)^2, (1 + R_2)^{-1} \leq \tilde{A} \leq R_1(1 + R_1)^{-1}, \tilde{\lambda} \in \Sigma_{\epsilon}\}. \end{aligned}$$

If (λ, ξ') satisfies the condition: $R_2^{-1}|\lambda|^{1/2} \leq A \leq R_1|\lambda|^{1/2}$ and $\lambda \in \Sigma_{\epsilon}$, then $(\tilde{\lambda}, \tilde{A}) \in D_{\epsilon}(R_1, R_2)$. Note that $\lambda \neq 0$ when $(\tilde{\lambda}, \tilde{A}) \in D_{\epsilon}(R_1, R_2)$. We define \tilde{L}_{ij} by replacing A_+ , A and B_{\pm} by \tilde{A}_+ , \tilde{A} and \tilde{B}_{\pm} in (3.16), respectively. And the matrix \tilde{L} is defined by replacing L_{ij}^{\pm} by \tilde{L}_{ij}^{\pm} in (3.18). Setting $\det \tilde{L} = \tilde{L}_{11}\tilde{L}_{22} - \tilde{A}\tilde{L}_{12}\tilde{L}_{21}$, we have

$$\det L = (|\lambda|^{1/2} + A)^4 \det \tilde{L}. \quad (5.12)$$

We prove that $\det \tilde{L} \neq 0$ provided that $(\tilde{\lambda}, \tilde{A}) \in D_{\epsilon}(R_1, R_2)$ by contradiction. Suppose that $\det \tilde{L} = 0$. By (5.12) $\det L = 0$, so that in view of (3.18) there exist $w_{\pm J}$ and p_- of the forms: $w_{\pm J}(x_N) = \alpha_{\pm J}(e^{\mp B_{\pm}x_N} - e^{\mp A_{\pm}x_N}) + \beta_{\pm J}e^{\mp B_{\pm}x_N}$ and $p_-(x_N) = \gamma_-e^{A_-x_N}$ with $A_- = A$ such that $\mathbf{w}_{\pm}(x_N) = (w_{\pm 1}(x_N), \dots, w_{\pm N}(x_N)) \neq (0, \dots, 0)$, and $\mathbf{w}_{\pm}(x_N)$ and $p_-(x_N)$ satisfy (3.2) and (3.3) with $\hat{h}_m(0) = 0$ and $\hat{H}(0) = 0$, that is they satisfy the following homogeneous equations:

$$\begin{aligned} \rho_+\lambda w_{+j} - \sum_{m=1}^{N-1} \mu_+ i\xi_m(i\xi_j w_{+m} + i\xi_m w_{+j}) - \mu_+ \partial_N(i\xi_j w_{+N} + \partial_N w_{+j}) \\ - (\nu_+ - \mu_+)i\xi_j(i\xi' \cdot w'_+ + \partial_N w_{+N}) &= 0 \\ \rho_+\lambda w_{+N} - \sum_{m=1}^{N-1} \mu_+ i\xi_m(\partial_N w_{+m} + i\xi_m w_{+N}) - 2\mu_+ \partial_N^2 w_{+N} & \end{aligned} \quad \text{for } x_N > 0,$$

$$\begin{aligned}
& -(\nu_+ - \mu_+) \partial_N (i\xi' \cdot w'_+ + \partial_N w_{+N}) = 0 && \text{for } x_N > 0, \\
& \rho_- \lambda w_{-j} - \sum_{m=1}^{N-1} \mu_- i\xi_m (i\xi_j w_{-m} + i\xi_m w_{-j}) - \mu_- \partial_N (i\xi_j w_{-N} + \partial_N w_j) + i\xi_j p_- = 0 && \text{for } x_N < 0, \\
& \rho_- \lambda w_{-N} - \sum_{m=1}^{N-1} \mu_- i\xi_m (\partial_N w_{-m} + i\xi_m w_{-N}) - 2\mu_- \partial_N^2 w_{-N} + \partial_N p_- = 0 && \text{for } x_N < 0, \\
& i\xi' \cdot w'_- + \partial_N w_{-N} = 0 && \text{for } x_N < 0, \\
& \mu_- (\partial_N w_{-j} + i\xi_j w_-)|_- - \mu_+ (\partial_N w_{+j} + i\xi_j w_{+N})|_+ = 0, \\
& (2\mu_- \partial_N w_{-N} - p_-)|_- - (2\mu_+ \partial_N w_{+N} + (\nu_+ - \mu_+) (i\xi' \cdot w'_+ + \partial_N w_{+N})|_+ = 0.
\end{aligned} \tag{5.13}$$

Set $(a, b)_\pm = \pm \int_0^{\pm\infty} a(x_N) \overline{b(x_N)} dx_N$ and $\|a\|_\pm = (a, a)_\pm^{1/2}$. Multiplying the equations in (5.13) by $\overline{w_{\pm j}}$ and using the integration by parts and the jump conditions in (5.13), we have

$$\begin{aligned}
0 &= \lambda (\rho_+ \sum_{m=1}^N \|w_{+m}\|^2 + \rho_- \sum_{m=1}^N \|w_{-m}\|^2) + \mu_+ [\sum_{j,k=1}^{N-1} \|i\xi_k w_{+j}\|_+^2 + \|i\xi' \cdot w'_+\|_+^2 + \sum_{j=1}^{N-1} \|\partial_N w_{+j}\|_+^2 \\
&+ \sum_{j=1}^{N-1} (i\xi_j w_{+N}, \partial_N w_{+N})_+ + \sum_{j=1}^{N-1} \|i\xi_j w_{+N}\|_+^2 + \sum_{j=1}^{N-1} (\partial_N w_{+j}, i\xi_j w_{+N})_+ + 2\|\partial_N w_{+N}\|_+^2] \\
&+ (\nu_+ - \mu_+) [\|i\xi' \cdot w'_+\|_+^2 + (\partial_N w_{+N}, i\xi' \cdot w'_+)_+ + (i\xi' \cdot w'_+, \partial_N w_{+N})_+ + \|\partial_N w_{+N}\|_+^2] \\
&+ \mu_- [\sum_{j,k=1}^{N-1} \|i\xi_k w_{-j}\|_-^2 + \|i\xi' \cdot w'_-\|_-^2 + \sum_{j=1}^{N-1} \|\partial_N w_{-j}\|_-^2 \\
&+ \sum_{j=1}^{N-1} (i\xi_j w_{-N}, \partial_N w_{-N})_- + \sum_{j=1}^{N-1} \|i\xi_j w_{-N}\|_-^2 + \sum_{j=1}^{N-1} (\partial_N w_{-j}, i\xi_j w_{-N})_- + 2\|\partial_N w_{-N}\|_-^2] \\
&= \lambda (\gamma_{0+} \|w_+\|_+^2 + \gamma_{0-} \|w_-\|_-^2) + \mu_+ [\sum_{j,k=1}^{N-1} \|i\xi_k w_{+j}\|_+^2 + \|i\xi' \cdot w'_+\|_+^2 + \sum_{j=1}^{N-1} \|\partial_N w_{+j} + i\xi_j w_{+N}\|_+^2 \\
&+ 2\|\partial_N w_{+N}\|_+^2] + (\nu_+ - \mu_+) \|\partial_N w_{+N} + i\xi' \cdot w'_+\|_+^2 \\
&+ \mu_- [\sum_{j,k=1}^{N-1} \|i\xi_k w_{-j}\|_-^2 + \|i\xi' \cdot w'_-\|_-^2 + \sum_{j=1}^{N-1} \|\partial_N w_{-j} + i\xi_j w_{-N}\|_-^2 + 2\|\partial_N w_{-N}\|_-^2].
\end{aligned} \tag{5.14}$$

Taking the real part and the imaginary part in (5.14), using the inequality:

$$\sum_{j,k=1}^{N-1} \|i\xi_j w_{+k}\|_+^2 + \|i\xi' \cdot w'_+\|_+^2 + 2\|\partial_N w_{+N}\|_+^2 \geq 2(\|i\xi' \cdot w'_+\|_+^2 + \|\partial_N w_{+N}\|_+^2) \geq \|\partial_N w_{+N} + i\xi' \cdot w'_+\|_+^2,$$

and setting $K = \rho_+ \|\mathbf{w}_+\|_+^2 + \rho_- \|\mathbf{w}_-\|_-^2$ and $L = \|\partial_N w_{+N} + i\xi' \cdot w'_+\|_+^2$ for short, we have

$$(\text{Im } \lambda) K = 0, \quad 0 \geq (\text{Re } \lambda) K + \nu_+ L. \tag{5.15}$$

When $\text{Im } \lambda \neq 0$, obviously $\mathbf{w}_\pm = 0$ which leads to a contradiction. When $\text{Im } \lambda = 0$, $\lambda \geq 0$ and $\lambda \neq 0$, because $\lambda \in \Sigma_\epsilon$. By (5.15) we have $L = 0$, because $\nu_+ > 0$. Thus, by (5.14) we have

$$\partial_N w_{\pm N} = 0, \quad \partial_N w_{\pm j} + i\xi_j w_{\pm N} = 0 \quad \text{on } \mathbb{R}^\pm \tag{5.16}$$

where $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{R}^- = (-\infty, 0)$. By the first equation of (5.16), $w_{\pm N}(x_N)$ are constants on \mathbb{R}^\pm , but $w_{\pm N}(x_N) \rightarrow 0$ as $\pm x_N \rightarrow \infty$, so that $w_{\pm N} = 0$. Thus, by the second equation of (5.16) we have $\partial_N w_{\pm j}(x_N) = 0$ on \mathbb{R}^\pm . But, $w_{\pm j}(x_N) \rightarrow 0$ as $\pm x_N \rightarrow \infty$, so that $w_{\pm j}(x_N) = 0$. Thus, we have $\mathbf{w}_\pm = 0$ when $\text{Im } \lambda = 0$, which leads to a contradiction. Therefore, we have $\det \tilde{L} \neq 0$ for $(\tilde{\lambda}, \tilde{A}) \in D_\epsilon(R_1, R_2)$. Since $D_\epsilon(R_1, R_2)$ is compact, we have

$$\inf_{(\tilde{\lambda}, \tilde{A}) \in D_\epsilon(R_1, R_2)} |\det \tilde{L}| = c > 0,$$

which, combined with (5.12), furnishes that

$$|\det L| \geq c(|\lambda|^{1/2} + A)^4 \quad (5.17)$$

provided that $R_2^{-1}|\lambda|^{1/2} \leq A \leq R_1|\lambda|^{1/2}$ and $\lambda \in \Sigma_\epsilon$. Setting $\omega = \min(c, \frac{1}{2}\omega_1, \frac{1}{2}\omega_2)$, by (5.7), (5.11) and (5.17), we have (5.1).

Next, we prove (5.2). Recalling (3.17) and the formula (5.3), by Lemma 3.2 we have

$$|\partial_{\xi'}^{\kappa'} \{(\tau \partial_\tau)^\ell \det L\}| \leq C_{\kappa'} (|\lambda|^{1/2} + A)^4 A^{-|\kappa'|} \quad (\ell = 0, 1) \quad (5.18)$$

for any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$ and $(\lambda, \xi') \in \tilde{\Sigma}_{\epsilon, 0}$. Thus, by the Bell formula (4.4) with $f(t) = 1/t$ and $g(\xi') = \det L$, (5.18) and (5.1), we have

$$|\partial_{\xi'}^{\kappa'} (\det L)^{-1}| \leq C_{\kappa'} \sum_{\ell=1}^{|\kappa'|} |\det L|^{-(\ell+1)} (|\lambda|^{1/2} + A)^{4\ell} A^{-|\kappa'|} \leq C_{\kappa'} (|\lambda|^{1/2} + A)^{-4} A^{-|\kappa'|},$$

which shows (5.2) with $\ell = 0$. Analogously, we have (5.2) with $\ell = 1$, which completes the proof of Lemma 5.1. \square

6 Problem with surface tension and height function

In this section, we consider the problem:

$$\begin{aligned} \rho_+ \lambda \mathbf{u}_+ - \operatorname{Div} \mathbf{S}_+(\mathbf{u}_+) &= 0 && \text{in } \mathbb{R}_+^N, \\ \rho_- \lambda \mathbf{u}_- - \operatorname{Div} \mathbf{S}_-(\mathbf{u}_-) + \nabla \pi_- &= 0, \quad \operatorname{div} \mathbf{u}_- = 0 && \text{in } \mathbb{R}_-^N, \\ \mu_- \mathbf{D}_{mN}(\mathbf{u}_-)|_- - \mu_+ \mathbf{D}_{mN}(\mathbf{u}_+)|_+ &= 0, \\ (\mu_- D_{NN}(\mathbf{u}_-) - \pi_-)|_- &= \sigma_- \Delta' H, \\ (\mu_+ \mathbf{D}_{NN}(\mathbf{u}_+) + (\nu_+ - \mu_+) \operatorname{div} \mathbf{u}_+)|_+ &= \sigma_+ \Delta' H, \\ u_{-m}|_- - u_{+m}|_+ &= 0, \\ \lambda H - \left(\frac{\rho_-}{\rho_- - \rho_+} u_{-N}|_- - \frac{\rho_+}{\rho_- - \rho_+} u_{+N}|_+ \right) &= d, \end{aligned} \quad (6.1)$$

where, $\sigma_\pm = \frac{\rho_\pm \sigma}{\rho_- - \rho_+}$, and prove the following theorem.

Theorem 6.1. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Then, there exist a $\lambda_0 > 0$ depending solely on μ_\pm , ν_+ , ρ_\pm and ϵ and operator families $\mathcal{U}_\pm(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(W_q^2(\mathbb{R}^N), W_q^2(\mathbb{R}_\pm^N)^N))$, $\mathcal{P}_{-3}(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(W_q^2(\mathbb{R}^N), \dot{W}_q^1(\mathbb{R}_-^N)))$ and $\mathcal{H}(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(W_q^2(\mathbb{R}^N), W_q^3(\mathbb{R}^N)))$ such that for any $\lambda \in \Sigma_{\epsilon, \lambda_0}$ and $d \in W_q^2(\mathbb{R}^N)$, problem (6.1) admits a unique solutions $\mathbf{u}_\pm = \mathcal{U}_\pm(\lambda)d$, $\pi_- = \mathcal{P}_{-3}(\lambda)d$ and $H = \mathcal{H}(\lambda)d$, and*

$$\begin{aligned} \mathcal{R}_{\mathcal{L}(W_q^2(\mathbb{R}^N), L_q(\mathbb{R}_\pm^N)^{N+N^2+N^3})}(\{(\tau \partial_\tau)^\ell (G_\lambda \mathcal{U}_\pm(\lambda)) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq \gamma \quad (\ell = 0, 1), \\ \mathcal{R}_{\mathcal{L}(W_q^2(\mathbb{R}^N), L_q(\mathbb{R}_-^N)^N)}(\{(\tau \partial_\tau)^\ell (\nabla \mathcal{P}_{-3}(\lambda)) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq \gamma \quad (\ell = 0, 1), \\ \mathcal{R}_{\mathcal{L}(W_q^2(\mathbb{R}^N), L_q(\mathbb{R}^N)^{N+1})}(\{(\tau \partial_\tau)^\ell ((\lambda, \nabla) \mathcal{H}(\lambda)) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq \gamma \quad (\ell = 0, 1) \end{aligned}$$

with some constant γ depending solely on μ_\pm , ν_+ , ρ_\pm and ϵ .

Remark 6.2. Combining Theorem 6.1 with Theorem 2.4, Theorem 2.6 and Theorem 2.8, we have Theorem 2.2 immediately.

As was discussed in Sect.3, applying the partial Fourier transform to (6.1), we have the equations (3.11) with interface condition:

$$\begin{cases} \mu_- (D_N \hat{u}_{-m} + i \xi_m \hat{u}_{-N})|_- - \mu_+ (D_N \hat{u}_{-m} + i \xi_m \hat{u}_{-N})|_+ = 0, \\ (2\mu_- D_N \hat{u}_{-N} - \hat{\pi}_-)|_- = -\sigma_- A^2 \hat{H}(0), \\ (2\mu_+ D_N \hat{u}_{-N} + (\nu_+ - \mu_+) (i \xi' \cdot \hat{u}'_+ + D_N \hat{u}_{+N})|_+ = -\sigma_+ A^2 \hat{H}(0), \\ \hat{u}_{-m}|_- - \hat{u}_{+m}|_+ = 0 \end{cases} \quad (6.2)$$

and the the resolvent equation for H :

$$\lambda \hat{H}(0) - \left(\frac{\rho_-}{\rho_- - \rho_+} \hat{u}_{-N}|_- - \frac{\rho_+}{\rho_- - \rho_+} \hat{u}_{+N}|_+ \right) = \hat{d}(0). \quad (6.3)$$

We look for solutions $\hat{u}_{\pm J}$ and \hat{p}_- of the form (3.7) with $h_j = 0$, so that especially $\hat{u}_{N,3}^\pm = 0$. Our task is to represent \hat{H} in terms of $\hat{d}(0)$. In view of (3.10), we have $\hat{u}_{\pm J}(0) = \beta_\pm$, so that by (3.22) with $\hat{h}_j(0) = 0$, we have

$$\begin{aligned} \beta_{+N} &= -\frac{A}{\det L} (\mathcal{L}_{22}\sigma_- A^2 + \mathcal{L}_{23}\sigma_+ A) \hat{H}(0), \\ \beta_{-N} &= -\frac{A}{\det L} (\mathcal{L}_{32}\sigma_- A^2 + \mathcal{L}_{33}\sigma_+ A) \hat{H}(0). \end{aligned} \quad (6.4)$$

Inserting these formulas into (6.3), we have

$$(\lambda + K) \hat{H}(0) = \hat{d}(0) \quad (6.5)$$

with

$$K = \frac{A^3}{\det L} \left(\frac{\rho_- \sigma_-}{\rho_- - \rho_+} \mathcal{L}_{32} - \frac{\rho_+ \sigma_-}{\rho_- - \rho_+} \mathcal{L}_{22} \right) + \frac{A^2}{\det L} \left(\frac{\rho_- \sigma_+}{\rho_- - \rho_+} \mathcal{L}_{33} - \frac{\rho_+ \sigma_+}{\rho_- - \rho_+} \mathcal{L}_{23} \right). \quad (6.6)$$

We prove that

Lemma 6.3. *Let $0 < \epsilon < \pi/2$ and let K be the function defined in (6.6). Then, there exists a positive constant λ_0 depending on ϵ , μ_\pm , ν_+ and ρ_\pm such that*

$$|\partial_{\xi'}^{\kappa'} ((\tau \partial_\tau)^\ell (\lambda + K)^{-1})| \leq C_{\kappa'} (|\lambda| + A)^{-1} A^{-|\kappa'|} \quad (\ell = 0, 1) \quad (6.7)$$

for any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$ and $(\lambda, \xi') \in \tilde{\Sigma}_{\epsilon, \lambda_0}$ with some constant $C_{\kappa'}$ depending on κ' , λ_0 , ϵ , μ_\pm , ν_+ and ρ_\pm .

Proof. To prove (6.7) with $\kappa' = 0$ and $\ell = 0$, first we consider the case where $R_1 |\lambda|^{1/2} \leq A$ with large R_1 . In the following, δ_1 is the same small number as in the proof of Lemma 5.1. We know the asymptotic formula for $\det L$ given in (5.6). On the other hand, by (5.4), (5.5) and (3.19), we have

$$\begin{aligned} \mathcal{L}_{32} &= (L_{11}^+ + L_{11}^-) L_{22}^+ - L_{12}^+ L_{21}^+ \\ &= \left\{ \left(\frac{2\mu_+(\mu_+ + \nu_+)}{2\mu_+ + \nu_+} + 2\mu_- \right) \frac{2\mu_+(\mu_+ + \nu_+)}{2\mu_+ + \nu_+} - \left(\frac{2(\mu_+)^2}{2\mu_+ + \nu_+} \right)^2 \right\} A^2 (1 + O(\delta_1)) \\ &= \frac{4\mu_+((\mu_+ + \mu_-)\nu_+ + \mu_+\mu_-)}{2\mu_+ + \nu_+} A^2 (1 + O(\delta_1)), \\ \mathcal{L}_{22} &= L_{21}^+ L_{12}^- = A^2 O(\delta_1), \quad \mathcal{L}_{33} = -L_{12}^+ L_{21}^- = A^3 O(\delta_1), \\ \mathcal{L}_{23} &= L_{12}^- L_{21}^- - (L_{11}^+ + L_{11}^-) L_{22}^- \\ &= -4 \left(\frac{\mu_+(\mu_+ + \nu_+)}{2\mu_+ + \nu_+} + \mu_- \right) \mu_- A^3 (1 + O(\delta_1)). \end{aligned}$$

Since $\frac{\rho_\pm \sigma_\pm}{\rho_- - \rho_+} = \left(\frac{\rho_\pm \sigma}{\rho_- - \rho_+} \right)^2$, we have

$$A^3 \left(\frac{\rho_- \sigma_-}{\rho_- - \rho_+} \mathcal{L}_{32} - \frac{\rho_+ \sigma_-}{\rho_- - \rho_+} \mathcal{L}_{22} \right) + A^2 \left(\frac{\rho_- \sigma_+}{\rho_- - \rho_+} \mathcal{L}_{33} - \frac{\rho_+ \sigma_+}{\rho_- - \rho_+} \mathcal{L}_{23} \right) = \omega_3 A^5 (1 + O(\delta_1))$$

with

$$\omega_3 = 4 \frac{\mu_+((\mu_+ + \mu_-)\nu_+ + \mu_+\mu_-)}{2\mu_+ + \nu_+} \left(\frac{\rho_-}{\rho_- - \rho_+} \right)^2 + 4 \left(\frac{\mu_+(\mu_+ + \nu_+)}{2\mu_+ + \nu_+} + \mu_- \right) \mu_- \left(\frac{\rho_+}{\rho_- - \rho_+} \right)^2.$$

Thus, by (5.6) we have

$$K = \frac{\omega_3}{\omega_1} A (1 + O(\delta_1)). \quad (6.8)$$

Since $\lambda \in \Sigma_\epsilon$, by (6.8) and (4.1) we have

$$|\lambda + K| \geq (\sin \frac{\epsilon}{2})(|\lambda| + \frac{\omega_3}{\omega_1}A) - \frac{\omega_3}{\omega_1}AO(\delta_1).$$

If we choose δ_1 so small that $O(\delta_1) \leq \frac{1}{2} \sin \frac{\epsilon}{2}$, we have

$$|\lambda + K| \geq (\frac{1}{2} \sin \frac{\epsilon}{2})(|\lambda| + \frac{\omega_3}{\omega_1}A) \quad (6.9)$$

provided that $R_1|\lambda|^{1/2} \leq A$ with large $R_1 > 0$ and $\lambda \in \Sigma_\epsilon$.

Next, we consider the case where $A \leq R_1|\lambda|^{1/2}$. By (5.1) we have

$$|\det L|^{-1} \leq \omega^{-1}(1 + R_1)^4|\lambda|^2 \quad (6.10)$$

for any $(\lambda, \xi') \in \tilde{\Sigma}_{\epsilon,0}$ provided that $A \leq R_1|\lambda|^{1/2}$. On the other hand, by (4.1) and (4.6), we have

$$|P(\lambda, \xi')| \leq \frac{C(|\lambda| + A^2)}{(\sin \frac{\epsilon}{2})(\rho_+(2\mu_+ + \nu_+)^{-1}|\lambda| + A^2)} \leq C. \quad (6.11)$$

Here and in the sequel, C denotes a generic constant depending on $R_1, \mu_\pm, \nu_\pm, \rho_\pm$ and ϵ . By (6.11) and (4.9), we have

$$|L_{11}^+| \leq C|\lambda|^{1/2}, \quad |L_{12}^+| \leq C|\lambda|, \quad |L_{21}^+| \leq C, \quad |L_{22}^+| \leq C|\lambda|^{1/2}. \quad (6.12)$$

Moreover, by the definition of L_{ij}^- given in (3.16) we have easily

$$|L_{11}^-| \leq C|\lambda|^{1/2}, \quad |L_{12}^-| \leq C|\lambda|, \quad |L_{21}^-| \leq C|\lambda|^{1/2}, \quad |L_{22}^-| \leq C|\lambda|,$$

which, combined with (6.10) and (6.12), furnishes that

$$|K| \leq C|\lambda|^{1/2} \quad (6.13)$$

for any $(\lambda, \xi') \in \tilde{\Sigma}_{\epsilon,0}$ provided that $A \leq R_1|\lambda|^{1/2}$. Thus, we have $|\lambda + K| \geq |\lambda|^{1/2}(|\lambda|^{1/2} - C)$, so that choosing $\lambda_0 > 0$ so large that $C\lambda_0^{1/2} \leq 1/2$, we have $|\lambda + K| \geq \frac{1}{2}|\lambda|$ for any $(\lambda, \xi') \in \tilde{\Sigma}_{\epsilon,\lambda_0}$ provided that $A \leq R_1|\lambda|^{1/2}$. Since $A \leq R_1|\lambda|^{1/2}$, we observe that

$$|\lambda + K| \geq \frac{1}{4}|\lambda| + \frac{1}{4}|\lambda| \geq \frac{1}{4}|\lambda| + \frac{\lambda_0^{1/2}}{4}|\lambda|^{1/2} \geq \frac{1}{4}(|\lambda| + \lambda_0^{1/2}R_1^{-1}A).$$

Choosing $R_1 > 0$ so large that $\lambda_0^{1/2}R_1^{-1} \leq \frac{\omega_3}{\omega_1}$, we have

$$|\lambda + K| \geq \frac{1}{4}(|\lambda| + \frac{\omega_3}{\omega_1}A) \quad (6.14)$$

for any $(\lambda, \xi') \in \tilde{\Sigma}_{\epsilon,0}$ provided that $A \leq R_1|\lambda|^{1/2}$. Since $\Sigma_{\epsilon,\lambda_0} \subset \Sigma_\epsilon$, combining (6.9) and (6.14), we have

$$|\lambda + K| \geq \omega_4(|\lambda| + A) \quad (6.15)$$

for any $(\lambda, \xi') \in \tilde{\Sigma}_{\epsilon,\lambda_0}$ with $\omega_4 = \min(\frac{1}{4}, \frac{1}{4}\frac{\omega_3}{\omega_1}, \frac{1}{2}\sin \frac{\epsilon}{2}, \frac{1}{2}\sin \frac{\epsilon}{2}\frac{\omega_3}{\omega_1})$.

Next, we prove (6.7) for any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$. By (3.17), (3.21) and Lemma 3.2, we have $K \in \mathbf{M}_{1,2}(0)$, so that by Bell's formula (4.4) with $f(t) = (\lambda + t)^{-1}$ and $g = K$, we have

$$|\partial_{\xi'}^{\kappa'}(\lambda + K)^{-1}| \leq C_{\kappa'} \sum_{\ell=1}^{|\kappa'|} |\lambda + K|^{-(\ell+1)} (|\lambda|^{1/2} + A)^\ell A^{-|\kappa'|} \leq C_{\kappa'} (|\lambda| + A)^{-1} A^{-|\kappa'|}.$$

Analogously, we have

$$|\partial_{\xi'}^{\kappa'}(\tau \partial_\tau(\lambda + K)^{-1})| \leq C_{\kappa'} (|\lambda| + A)^{-1} A^{-|\kappa'|}.$$

Summing up, we have obtained (6.7). This completes the proof of Lemma 6.3. \square

By (6.5) and Lemma 6.3, we have

$$\hat{H}(\xi', 0) = (\lambda + K)^{-1} \hat{d}(\xi', 0), \quad (6.16)$$

so that we define $\hat{H}(\xi', x_N)$ by $\hat{H}(\xi', x_N) = e^{-(1+A^2)^{1/2}x_N}(\lambda + K)^{-1} \hat{d}(\xi', 0)$. We have the following lemma.

Lemma 6.4. . Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and let λ_0 be the same constant as in Lemma 6.3. Given that the operator $\tilde{\mathcal{H}}(\lambda)$ is defined by

$$[\tilde{\mathcal{H}}(\lambda)d](x) = \mathcal{F}_{\xi'}^{-1}[e^{-(1+A^2)^{1/2}x_N}(\lambda + K)^{-1} \hat{d}(\xi', 0)](x') \quad \text{for } x \in \mathbb{R}_+^N \quad (6.17)$$

for any $d \in W_q^2(\mathbb{R}^N)$, $\tilde{\mathcal{H}}(\lambda) \in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(W_q^2(\mathbb{R}^N), W_q^3(\mathbb{R}_+^N)))$, and

$$\mathcal{R}_{\mathcal{L}(W_q^2(\mathbb{R}^N), W_q^2(\mathbb{R}_+^{1+N}))}(\{(\tau \partial_\tau)^\ell(\lambda, \nabla) \tilde{\mathcal{H}}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) \leq \gamma \quad (\ell = 0, 1) \quad (6.18)$$

with some constant γ depending on μ_\pm , ν_+ , ρ_\pm , ϵ and λ_0 .

Proof. First, using the same idea as in (3.27) and (3.28) and the identity: $1 = \frac{1+A^2}{1+A^2} = \frac{1}{1+A^2} - \sum_{k=1}^{N-1} \frac{(i\xi_k)(i\xi_k)}{1+A^2}$, we rewrite $\tilde{\mathcal{H}}(\lambda)d$ as follows:

$$\begin{aligned} \tilde{\mathcal{H}}(\lambda)d &= \mathcal{F}_{\xi'}^{-1} \left[\frac{e^{-(1+A^2)^{1/2}x_N}}{\lambda + K} \hat{d}(\xi', 0) \right] = - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{\partial}{\partial y_N} \frac{e^{-(1+A^2)^{1/2}(x_N+y_N)}}{\lambda + K} \hat{d}(\xi', y_N) \right] dy_N \\ &= - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{e^{-(1+A^2)^{1/2}(x_N+y_N)}}{(\lambda + K)(1+A^2)} \widehat{\partial_N d}(\xi', y_N) \right] (x') dy_N \\ &\quad + \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{e^{-(1+A^2)^{1/2}(x_N+y_N)}(i\xi_k)}{(\lambda + K)(1+A^2)} \widehat{\partial_k \partial_N d}(\xi', y_N) \right] (x') dy_N \\ &\quad + \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{e^{-(1+A^2)^{1/2}(x_N+y_N)}}{(\lambda + K)(1+A^2)^{1/2}} (1 - \Delta') \widehat{d}(\xi', y_N) \right] (x') dy_N \end{aligned} \quad (6.19)$$

for $x_N > 0$.

To prove the \mathcal{R} boundedness of the operator family $\{\tilde{\mathcal{H}}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}$, we prepare the following three lemmas.

Lemma 6.5. Let $1 < q < \infty$ and let $\ell(\xi')$ be a C^∞ function defined on \mathbb{R}^{N-1} satisfying the estimates:

$$|\partial_{\xi'}^{\kappa'} \ell(\xi')| \leq C_{\kappa'} (1 + A)^{1-|\kappa'|}$$

for any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$ and $\xi' \in \mathbb{R}^{N-1} \setminus \{0\}$. Let T be the operator defined by

$$Tf = \int_0^\infty \mathcal{F}_{\xi'}^{-1} [e^{-(1+A^2)^{1/2}(x_N+y_N)} \ell(\xi') \hat{f}(\xi', y_N)] (x') dy_N$$

for $f \in L_q(\mathbb{R}_+^N)$. Then, $T \in \mathcal{L}(L_q(\mathbb{R}_+^N))$.

Proof. If we define a function $k(x)$ by $k(x) = \mathcal{F}_{\xi'}^{-1} [e^{-(1+A^2)^{1/2}x_N} \ell(\xi')](x')$, we have

$$[Tf](x) = \int_0^\infty \int_{\mathbb{R}^{N-1}} k(x' - y', x_N + y_N) f(y', y_N) dy' dy_N.$$

Our task is to prove that

$$|k(x)| \leq C|x|^{-N}. \quad (6.20)$$

In fact, if we have (6.20), then by Young's inequality we have

$$\|[Tf](\cdot, x_N)\|_{L_q(\mathbb{R}^{N-1})} \leq \int_0^\infty \|k(\cdot, x_N + y_N)\|_{L_1(\mathbb{R}^{N-1})} \|f(\cdot, y_N)\|_{L_q(\mathbb{R}^{N-1})} dy_N.$$

By (6.20), we have $\|k(\cdot, x_N + y_N)\|_{L_1(\mathbb{R}^{N-1})} \leq C(x_N + y_N)^{-1}$, so that

$$\|[Tf](\cdot, x_N)\|_{L_q(\mathbb{R}^{N-1})} \leq C \int_0^\infty \frac{\|f(\cdot, y_N)\|_{L_q(\mathbb{R}^{N-1})}}{x_N + y_N} dy_N.$$

By the Hardy inequality (cf. Stein [15, p.271 A.3]), we have $\|Tf\|_{L_q(\mathbb{R}_+^N)} \leq C\|f\|_{L_q(\mathbb{R}_+^N)}$. Thus, we have proved $T \in \mathcal{L}(L_q(\mathbb{R}_+^N))$.

To prove (6.20), first we observe that

$$k(x) = \sum_{|\alpha'|=N-1} \left(\frac{ix'}{|x'|^2} \right)^{\alpha'} \int_{\mathbb{R}^{N-1}} e^{ix' \cdot \xi'} \partial_{\xi'}^{\alpha'} (e^{-(1+A^2)^{1/2}x_N} \ell(\xi')) d\xi'. \quad (6.21)$$

For any multi-index $\beta' \in \mathbb{N}_0^{N-1}$, we have

$$\begin{aligned} |\partial_{\xi'}^{\beta'} \{(\partial_{\xi'}^{\alpha'} e^{-(1+A^2)^{1/2}x_N} \ell(\xi'))\}| &\leq C_{\alpha', \beta'} e^{-c_{\alpha', \beta'}(1+A^2)^{1/2}x_N} (1+A)^{1-|\alpha'|-|\beta'|} \\ &= C_{\alpha', \beta'} e^{-c_{\alpha', \beta'}(1+A^2)^{1/2}x_N} (1+A)^{2-N-|\beta'|} \end{aligned}$$

with some positive constants $C_{\alpha', \beta'}$ and $c_{\alpha', \beta'}$. Since $2-N \leq 0$, we have

$$|\partial_{\xi'}^{\beta'} \{(\partial_{\xi'}^{\alpha'} e^{-(1+A^2)^{1/2}x_N} \ell(\xi'))\}| \leq C_{\alpha', \beta'} e^{-c_{\alpha', \beta'}(1+A^2)^{1/2}x_N} A^{2-N-|\beta'|}$$

for any multi-index $\beta' \in \mathbb{N}_0^{N-1}$. Thus, by the result due to Shibata and Shimizu [12], we have

$$\left| \int_{\mathbb{R}^{N-1}} e^{ix' \cdot \xi'} \partial_{\xi'}^{\alpha'} (e^{-(1+A^2)^{1/2}x_N} \ell(\xi')) d\xi' \right| \leq C|x'|^{-(N-1+(2-N))} = C|x'|^{-1},$$

which, combined with (6.21), furnishes that

$$|k(x)| \leq C|x'|^{-N}. \quad (6.22)$$

On the other hand,

$$\begin{aligned} |k(x)| &\leq \int_{\mathbb{R}^{N-1}} e^{-(1+A^2)^{1/2}x_N} |\ell(\xi')| d\xi' \leq C \int_{\mathbb{R}^{N-1}} e^{-(1+A^2)^{1/2}x_N} (1+A) d\xi' \\ &\leq C \int_{\mathbb{R}^{N-1}} \frac{d\xi'}{((1+A^2)^{1/2}x_N)^N} + C \int_{\mathbb{R}^{N-1}} e^{-Ax_N} A d\xi' \\ &\leq \frac{C}{(x_N)^N} \int_0^\infty \frac{r^{N-2}}{(1+r^2)^{\frac{N}{2}}} dr + \frac{C}{(x_N)^N} \int_{\mathbb{R}^{N-1}} e^{-|\eta'|} |\eta'| d\eta' \leq \frac{C}{(x_N)^N}, \end{aligned}$$

which, combined with (6.22), furnishes (6.20). This completes the proof of Lemma 6.5. \square

The following lemma was proved in Enomoto-Shibata [2]

Lemma 6.6. *Let $1 < q < \infty$ and let Λ be a subset of \mathbb{C} . Let $m(\lambda, \xi')$ be a function defined on $\Lambda \times (\mathbb{R}^{N-1} \setminus \{0\})$ such that for any multi-index $\alpha \in \mathbb{N}_0^N$ there exists a constant $C_{\alpha'}$ depending on α' and Λ such that*

$$|\partial_{\xi'}^{\alpha'} m(\lambda, \xi')| \leq C_{\alpha'} A^{-|\alpha'|} \quad (6.23)$$

for any $(\lambda, \xi') \in \Lambda \times (\mathbb{R}^{N-1} \setminus \{0\})$. Let K_λ be an operator defined by $K_\lambda f = \mathcal{F}_{\xi'}^{-1}[m(\lambda, \xi') \hat{f}(\xi')]$. Then, the set $\{K_\lambda \mid \lambda \in \Lambda\}$ is \mathcal{R} -bounded on $\mathcal{L}(L_q(\mathbb{R}^{N-1}))$ and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^{N-1}))}(\{K_\lambda \mid \lambda \in \Lambda\}) \leq C_{q,N} \max_{|\alpha'| \leq N} C_{\alpha'} \quad (6.24)$$

with some constant $C_{q,N}$ that depends solely on q and N .

From the definition of \mathcal{R} boundedness we have the following lemma immediately.

Lemma 6.7. *Let $1 < q < \infty$ and let Λ be a subset of \mathbb{C} . Let $\{\mathcal{S}_\lambda \mid \lambda \in \Lambda\}$ be an \mathcal{R} bounded operator family on $\mathcal{L}(L_q(\mathbb{R}^{N-1}))$ and T a bounded linear operator in $\mathcal{L}(L_q(\mathbb{R}_+^N))$. Then, $\{\mathcal{S}_\lambda T \mid \lambda \in \Lambda\}$ is an \mathcal{R} -bounded operator family on $\mathcal{L}(L_q(\mathbb{R}_+^N))$ and*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_+^N))}(\{\mathcal{S}_\lambda T \mid \lambda \in \Lambda\}) \leq \mathcal{R}(\mathcal{S}_\lambda) \|T\|_{\mathcal{L}(L_q(\mathbb{R}_+^N))} \quad (6.25)$$

where we have set $\mathcal{R}(\mathcal{S}_\lambda) = \mathcal{R}_{L_q(\mathbb{R}^{N-1})}(\{\mathcal{S}_\lambda \mid \lambda \in \Lambda\})$ for short.

Under these preparations, we finish proving Lemma 6.4. For any multi-index $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq 2$, using (6.19), we write

$$\begin{aligned} \partial_x^\alpha(\lambda, \nabla) \tilde{\mathcal{H}}(\lambda) d &= - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{e^{-(1+A^2)^{1/2}(x_N+y_N)} \ell_\alpha(\xi')}{(1+A^2)} \frac{(\lambda, i\xi', -(1+A^2)^{1/2})}{(\lambda+K)} \widehat{\partial_N d}(\xi', y_N) \right] (x') dy_N \\ &+ \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{e^{-(1+A^2)^{1/2}(x_N+y_N)} (i\xi_k) \ell_\alpha(\xi')}{(1+A^2)} \frac{(\lambda, i\xi', -(1+A^2)^{1/2})}{(\lambda+K)} \widehat{\partial_k \partial_N d}(\xi', y_N) \right] (x') dy_N \\ &+ \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{e^{-(1+A^2)^{1/2}(x_N+y_N)} \ell_\alpha(\xi')}{(1+A^2)^{1/2}} \frac{(\lambda, i\xi', -(1+A^2)^{1/2})}{(\lambda+K)} (1-\Delta') d(\xi', y_N) \right] (x') dy_N \end{aligned}$$

where $\ell_\alpha(\xi')$ is an symbol satisfying the estimate:

$$|\partial_{\xi'}^{\kappa'} \ell_\alpha(\xi')| \leq C_{\kappa'} (1+A)^{|\alpha|-|\kappa'|}$$

for any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$. If we define the operators \mathcal{S}_λ , T_1^α , T_{2k}^α , and T_3^α by

$$\begin{aligned} [\mathcal{S}_\lambda g](x') &= \mathcal{F}_{\xi'}^{-1} \left[\frac{(\lambda, i\xi', -(1+A^2)^{1/2})}{(\lambda+K)} \hat{g}(\xi') \right] (x'), \\ [T_1^\alpha f](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{e^{-(1+A^2)^{1/2}(x_N+y_N)} \ell_\alpha(\xi')}{(1+A^2)} \hat{f}(\xi', y_N) \right] (x') dy_N, \\ [T_{2k}^\alpha f](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{e^{-(1+A^2)^{1/2}(x_N+y_N)} (i\xi_k) \ell_\alpha(\xi')}{(1+A^2)} \hat{f}(\xi', y_N) \right] (x') dy_N, \\ [T_3^\alpha f](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{e^{-(1+A^2)^{1/2}(x_N+y_N)} \ell_\alpha(\xi')}{(1+A^2)^{1/2}} \hat{f}(\xi', y_N) \right] (x') dy_N \end{aligned}$$

for $g \in L_q(\mathbb{R}^{N-1})$ and $f \in L_q(\mathbb{R}_+^N)$, then, we have

$$\partial_x^\alpha(\lambda, \nabla) \tilde{\mathcal{H}}(\lambda) d = -\mathcal{S}_\lambda T_1^\alpha (\partial_N d) + \sum_{k=1}^{N-1} \mathcal{S}_\lambda T_{2k}^\alpha (\partial_k \partial_N d) + \mathcal{S}_\lambda T_3^\alpha ((1-\Delta') d).$$

Since

$$\begin{aligned} |\partial_{\xi'}^{\kappa'} \left(\frac{\ell_\alpha(\xi')}{1+A^2} \right)| &\leq C_{\kappa'} (1+A)^{|\alpha|-2-|\kappa'|}, \quad |\partial_{\xi'}^{\kappa'} \left(\frac{\ell_\alpha(\xi') (i\xi_k)}{1+A^2} \right)| \leq C_{\kappa'} (1+A)^{|\alpha|-1-|\kappa'|}, \\ |\partial_{\xi'}^{\kappa'} \left(\frac{\ell_\alpha(\xi')}{(1+A^2)^{1/2}} \right)| &\leq C_{\kappa'} (1+A)^{|\alpha|-1-|\kappa'|}, \end{aligned} \quad (6.26)$$

for any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$ with some constant $C_{\kappa'}$, by Lemma 6.5 T_1^α , T_{2k}^α and T_3^α are bounded linear operators from $L_q(\mathbb{R}_+^N)$ into itself. By Lemma 6.3, we have

$$|\partial_{\xi'}^{\kappa'} ((\tau \partial_\tau)^\ell \frac{(\lambda, i\xi', -(1+A^2)^{1/2})}{(\lambda+K)})| \leq C_{\kappa'} A^{-|\kappa'|} \quad (\ell = 0, 1),$$

so that by Lemma 6.6 $(\tau \partial_\tau)^\ell \mathcal{S}_\lambda$ ($\ell = 0, 1$) are \mathcal{R} bounded operator families on $L_q(\mathbb{R}^{N-1})$ and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^{N-1}))}(\{(\tau \partial_\tau)^\ell \mathcal{S}_\lambda \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) \leq \gamma \quad (\ell = 0, 1)$$

with some constant γ depending solely on μ_\pm , ν_+ , ρ_\pm , ϵ and λ_0 . Thus, Lemma 6.4 follows from Lemma 6.7 immediately. This completes the proof of Lemma 6.4. \square

By using the Lions method, we extend $\tilde{\mathcal{H}}(\lambda)d$ to $x_N < 0$. Namely, we define $\mathcal{H}(\lambda)$ by

$$[\mathcal{H}(\lambda)d](x) = \begin{cases} [\tilde{\mathcal{H}}(\lambda)d](x) & (x_N > 0) \\ \sum_{j=1}^4 a_j [\tilde{\mathcal{H}}(\lambda)d](x', -jx_N) & (x_N < 0) \end{cases}$$

where a_j are constants satisfying the equations: $\sum_{j=1}^4 a_j (-j)^k = 1$ for $k = 0, 1, 2, 3$. By Lemma 6.4, we have the following corollary of Lemma 6.4 immediately.

Corollary 6.8. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and let λ_0 be the same constant as in Lemma 6.3. Then, there exists an operator family $\mathcal{H}(\lambda) \in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(W_q^2(\mathbb{R}^N), W_q^3(\mathbb{R}^N)))$ such that for any $\lambda \in \Sigma_{\epsilon, \lambda_0}$ and $d \in W_q^2(\mathbb{R}^N)$, $\mathcal{F}_{\xi'}^{-1} \left[\frac{\hat{d}(\xi', 0)}{\lambda + K} \right](x') = \mathcal{H}(\lambda)d|_{x_N=0}$ and*

$$\mathcal{R}_{\mathcal{L}(W_q^2(\mathbb{R}^N), W_q^2(\mathbb{R}^N)^{1+N})}(\{(\tau \partial_\tau)^\ell(\lambda, \nabla) \mathcal{H}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) \leq \gamma \quad (\ell = 0, 1)$$

with some constant γ depending on μ_\pm , ν_+ , ρ_\pm , ϵ and λ_0 .

Next, we give a solution operator of (6.1) for velocity field. By (3.7) and (6.5), we have

$$\hat{u}_{\pm J} = AM_\pm(x_N) \frac{AR_{JN,0}^\pm}{\lambda + K} \hat{d}(0) + Ae^{\mp B_\pm x_N} \frac{AS_{JN,-1}^\pm}{\lambda + K} \hat{d}(0), \quad p_- = e^{Ax_N} \frac{Ap_{N,1}^-}{\lambda + K} \hat{d}(0).$$

Employing the same argument as in (3.27) and (3.28) and using (3.30) and the identity: $1 = \frac{1+A^2}{1+A^2} = \frac{1}{1+A^2} - \sum_{k=1}^{N-1} \frac{(i\xi_k)(i\xi_k)}{1+A^2}$, we have

$$\begin{aligned} u_{\pm J}(x) &= \mathcal{F}_{\xi'}^{-1}[\hat{u}_{\pm J}(\xi', x_N)](x') \\ &= - \int_0^{\pm\infty} \mathcal{F}_{\xi'}^{-1} \left[\partial_N (AM_\pm(x_N + y_N) \frac{AR_{JN,0}^\pm}{\lambda + K} \hat{d}(\xi', y_N)) \right](x') dy_N \\ &\quad - \int_0^{\pm\infty} \mathcal{F}_{\xi'}^{-1} \left[\partial_N (Ae^{\mp B_\pm(x_N + y_N)} \frac{AS_{JN,-1}^\pm}{\lambda + K} \hat{d}(\xi', y_N)) \right](x') dy_N \\ &= - \int_0^{\pm\infty} \mathcal{F}_{\xi'}^{-1} \left[AM_\pm(x_N + y_N) \left\{ \frac{AR_{JN,0}^\pm}{(\lambda + K)(1 + A^2)} \widehat{\partial_N d}(\xi', y_N) \right. \right. \\ &\quad \left. \left. - \sum_{k=1}^{N-1} \frac{AR_{JN,0}^\pm(i\xi_k)}{(\lambda + K)(1 + A^2)} \widehat{\partial_k \partial_N d}(\xi', y_N) \mp \frac{A_\pm AR_{JN,0}^\pm}{(\lambda + K)(1 + A^2)} (1 - \widehat{\Delta'}) d(\xi', y_N) \right\} \right](x') dy_N \\ &\quad - \int_0^{\pm\infty} Ae^{\mp B_\pm(x_N + y_N)} \left\{ \mp \frac{AR_{JN,0}^\pm}{(\lambda + K)(1 + A^2)} (1 - \widehat{\Delta'}) d(\xi', y_N) + \frac{AS_{JN,-1}^\pm}{(\lambda + K)(1 + A^2)} \widehat{\partial_N d}(\xi', y_N) \right. \\ &\quad \left. - \sum_{k=1}^{N-1} \frac{AS_{JN,-1}^\pm(i\xi_k)}{(\lambda + K)(1 + A^2)} \widehat{\partial_k \partial_N d}(\xi', y_N) \mp \frac{B_\pm AS_{JN,-1}^\pm}{(\lambda + K)(1 + A^2)} (1 - \widehat{\Delta'}) d(\xi', y_N) \right\} \right](x') dy_N \end{aligned}$$

Analogously, we have

$$p_-(x) = \int_{-\infty}^0 \mathcal{F}_{\xi'}^{-1} \left[e^{A(x_N + y_N)} \frac{p_{N,1}^-}{\lambda + K} \left\{ (1 - \widehat{\Delta'}) d(\xi', y_N) - \sum_{k=1}^{N-1} \widehat{\partial_k \partial_N d}(\xi', y_N) \right\} \right](x') dy_N.$$

By (3.8), (4.5), Lemma 6.3 and (6.26) with $\alpha = 0$, we have

$$\begin{aligned} \frac{p_{N,1}^-}{\lambda + K} &\in \mathbf{M}_{0,2}(\lambda_0), \quad \frac{AR_{JN,0}^\pm(i\xi_k)}{(\lambda + K)(1 + A^2)}, \quad \frac{A_\pm AR_{JN,0}^\pm}{(\lambda + K)(1 + A^2)} \in \mathbf{M}_{-1,2}(\lambda_0), \\ \frac{AR_{JN,0}^\pm}{(\lambda + K)(1 + A^2)}, \quad \frac{AS_{JN,-1}^\pm}{(\lambda + K)(1 + A^2)}, \quad \frac{AS_{JN,-1}^\pm(i\xi_k)}{(\lambda + K)(1 + A^2)}, \quad \frac{B_\pm AS_{JN,-1}^\pm}{(\lambda + K)(1 + A^2)} &\in \mathbf{M}_{-2,2}(\lambda_0). \end{aligned} \tag{6.27}$$

In fact, by Leibniz's rule we have

$$|\partial_{\xi'}^{\alpha'} \left(\frac{A + AR_{JN,0}^+}{(\lambda + K)(1 + A^2)} \right)| \leq C_{\alpha'} \frac{|\lambda|^{1/2} + A}{(|\lambda| + A)(1 + A)} A^{-|\alpha'|}.$$

Since $(|\lambda| + A)(1 + A) = |\lambda| + |\lambda|A + A + A^2 \geq \frac{1}{2}(|\lambda|^{1/2} + A)^2$, we have $\frac{A \pm AR_{JN,0}^+}{(\lambda + K)(1 + A^2)} \in \mathbf{M}_{-1,2}(\lambda_0)$. Analogously, we have other assertions in (6.27). Therefore, by Lemma 3.4, Lemma 3.5 and Corollary 6.8 we have Theorem 6.1, which completes the proof of Theorem 2.2.

7 A proof of theorem 1.1

First, we transfer problem (1.4), (1.10) and (1.6) to the zero initial data case. Let e^{At} be the operator defined by $e^{At}f = \mathcal{F}_{\xi}^{-1}[e^{-(1+|\xi|^2)^{1/2}t} CF[f](\xi)]$. Here and in the sequel, $\mathcal{F}[f](\xi)$ and \mathcal{F}^{-1} denote the Fourier transform f on \mathbb{R}^N and the inverse transform of $g(\xi)$ defined by

$$\mathcal{F}[f](\xi) = \int_{\mathbb{R}^N} e^{-ix \cdot \xi} f(x) dx, \quad \mathcal{F}_{\xi}^{-1}[g(\xi)] = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{ix \cdot \xi} g(\xi) d\xi,$$

respectively. Since

$$\begin{aligned} \|\partial_t e^{At} f(\cdot, t)\|_{L_q(\mathbb{R}^N)} &\leq C t^{-1} e^{-t/2} \|f\|_{L_q(\mathbb{R}^N)}, \\ \|\partial_t e^{At} f(\cdot, t)\|_{L_q(\mathbb{R}^N)} &\leq C e^{-t/2} \|f\|_{W_q^1(\mathbb{R}^N)}, \\ \|e^{At} f(\cdot, t)\|_{L_q(\mathbb{R}^N)} &\leq C e^{-t/2} \|f\|_{L_q(\mathbb{R}^N)} \end{aligned}$$

for $1 < q < \infty$ and $t > 0$, employing the same real interpolation argument as in the proof of Theorem 3.9 in [13], we have

$$\|\partial_t e^{At} f\|_{L_p((0,\infty), L_q(\mathbb{R}^N))} + \|e^{At} f\|_{L_p((0,\infty), W_q^1(\mathbb{R}^N))} \leq C \|f\|_{B_{q,p}^{1-1/p}(\mathbb{R}^N)}.$$

Thus, setting $I = e^{At} H_0$, we have $I|_{t=0} = H_0$ in \mathbb{R}^N , and

$$\|\partial_t I\|_{L_p((0,\infty), W_q^2(\mathbb{R}^N))} + \|I\|_{L_p((0,\infty), W_q^3(\mathbb{R}^N))} \leq C \|H_0\|_{B_{q,p}^{3-1/p}(\mathbb{R}^N)}, \quad (7.1)$$

where $1 < p, q < \infty$. On the other hand, let $\tilde{\mathbf{u}}_{0\pm}$ be the extension of $\mathbf{u}_{0\pm}$ to \mathbb{R}_{\mp}^N such that $\tilde{\mathbf{u}}_{0\pm} = \mathbf{u}_{0\pm}$ on \mathbb{R}_{\pm}^N and

$$\|\tilde{\mathbf{u}}_{0\pm}\|_{B_{q,p}^{2(1-1/p)}(\mathbb{R}^N)} \leq C \|\mathbf{u}_{0\pm}\|_{B_{q,p}^{2(1-1/p)}(\mathbb{R}_{\pm}^N)}. \quad (7.2)$$

Setting

$$\mathbf{v}_{\pm} = e^{-(1-\Delta)t} \tilde{\mathbf{u}}_{0\pm} = \mathcal{F}_{\xi}^{-1}[e^{-(1+|\xi|^2)t} \mathcal{F}[\tilde{\mathbf{u}}_{0\pm}](\xi)],$$

obviously we see that $\mathbf{v}_{\pm}|_{t=0} = \mathbf{u}_{0\pm}$ in \mathbb{R}_{\pm}^N , and moreover by the same real interpolation argument in the proof of Theorem 3.9 in [13], we have

$$\|\partial_t \mathbf{v}_{\pm}\|_{L_p((0,\infty), L_q(\mathbb{R}^N))} + \|\mathbf{v}_{\pm}\|_{L_p((0,\infty), W_q^2(\mathbb{R}^N))} \leq C \|\mathbf{u}_{0\pm}\|_{B_{q,p}^{2(1-1/p)}(\mathbb{R}_{\pm}^N)}. \quad (7.3)$$

Setting $\mathbf{u}_{\pm} = \mathbf{v}_{\pm} + \mathbf{w}_{\pm}$ and $H = I + J$ in (1.4) and (1.10), we have

$$\begin{aligned} \rho_{*+} \partial_t \mathbf{w}_{+} - \operatorname{Div} \mathbf{S}_{*+}(\mathbf{w}_{+}) &= \tilde{\mathbf{f}}_{+} && \text{in } \mathbb{R}_{+}^N \times (0, T) \\ \rho_{*-} \partial_t \mathbf{w}_{-} - \operatorname{Div} \mathbf{S}_{*-}(\mathbf{w}_{-}) + \nabla p_{-} &= \tilde{\mathbf{f}}_{-}, \quad \operatorname{div} \mathbf{w}_{-} = \tilde{f}_{\operatorname{div}} = \operatorname{div} \tilde{\mathbf{f}}_{\operatorname{div}} && \text{in } \mathbb{R}_{-}^N \times (0, T) \\ \mu_{*-} D_{iN}(\mathbf{w}_{-})|_{-} - \mu_{*+} D_{iN}(\mathbf{w}_{+})|_{+} &= \tilde{g}_i \quad (i = 1, \dots, N-1) && \text{in } \mathbb{R}_0^N \times (0, T), \\ (\mu_{*-} D_{NN}(\mathbf{w}_{-}) - p_{-})|_{-} &= \frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}} \Delta' J && \text{in } \mathbb{R}_0^N \times (0, T), \\ (\mu_{*+} D_{NN}(\mathbf{w}_{+}) + (\nu_{*+} - \mu_{*+}) \operatorname{div} \mathbf{w}_{+} \mathbf{I})|_{+} &= \frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}} \Delta' J + \tilde{g}_{N+1} && \text{in } \mathbb{R}_0^N \times (0, T), \end{aligned}$$

$$\begin{aligned}
w_{-i}|_- - w_{+i}|_+ &= \tilde{h}_i \quad (i = 1, \dots, N-1) && \text{in } \mathbb{R}_0^N \times (0, T), \\
\partial_t J - \left(\frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}} w_{-N}|_- - \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}} w_{+N}|_+ \right) &= \tilde{d} && \text{in } \mathbb{R}_0^N \times (0, T), \\
\mathbf{w}_{\pm}|_{t=0} &= 0 \quad \text{in } \mathbb{R}_{\pm}^N, \quad J|_{t=0} = 0 \quad \text{in } \mathbb{R}^N && (7.4)
\end{aligned}$$

with

$$\begin{aligned}
\tilde{\mathbf{f}}_+ &= \mathbf{f}_+ - (\rho_{*+} \partial_t \mathbf{v}_+ - \text{Div } \mathbf{S}_{*+}(\mathbf{v}_+)), \quad \tilde{\mathbf{f}}_- = \mathbf{f}_- - (\rho_{*-} \partial_t \mathbf{v}_- - \text{Div } \mathbf{S}_{*-}(\mathbf{v}_-)) + \nabla \tilde{g}_N, \\
p_- &= \pi_- + \tilde{g}_N, \quad \tilde{g}_N = \frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}} (\sigma \Delta' H_1 + g_N - \rho_{*+} g_{N+1}) - \mu_{*-} D_{NN}(\mathbf{v}_-), \\
\tilde{f}_{\text{div}} &= f_{\text{div}} - \text{div } \mathbf{v}_-, \quad \tilde{\mathbf{f}}_{\text{div}} = \mathbf{f}_{\text{div}} - \mathbf{v}_-, \\
\tilde{g}_i &= g_- - (\mu_{*-} D_{iN}(\mathbf{v}_-)|_- - \mu_{*+} D_{iN}(\mathbf{v}_+)|_+) \quad (i = 1, \dots, N-1), \\
\tilde{g}_{N+1} &= \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}} (\sigma \Delta' H + g_N - \rho_{*-} g_{N+1}) - (\mu_{*+} D_{NN}(\mathbf{v}_+) + (\nu_{*+} - \mu_{*+}) \text{div } \mathbf{v}_+)|_+, \\
\tilde{H}_- &= h_- - (v_{-i}|_- - v_{+i}|_+) \quad (i = 1, \dots, N-1), \\
\tilde{d} &= d + \left(\frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}} v_{-N}|_- - \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}} v_{+N}|_+ \right).
\end{aligned}$$

Setting

$$\begin{aligned}
\tilde{\mathbb{F}}_{p,q}(t) &= \|\tilde{\mathbf{f}}_+\|_{L_p((0,t), L_q(\mathbb{R}_+^N))} + \|\tilde{\mathbf{f}}_-\|_{L_p((0,t), L_q(\mathbb{R}_-^N))} + \|\tilde{f}_{\text{div}}\|_{L_p((0,t), W_q^1(\mathbb{R}^N))} + \|\tilde{f}_{\text{div}}\|_{L_p((0,t), W_q^{-1}(\mathbb{R}_-^N))} \\
&+ \|\partial_t \tilde{\mathbf{f}}_{\text{div}}\|_{L_p((0,T), L_q(\mathbb{R}_-^N))} + \sum_{i=1}^{N+1} (\|\tilde{g}_i\|_{L_p((0,t), W_q^1(\mathbb{R}^N))} + \|\partial_t \tilde{g}_i\|_{L_p((0,t), W_q^{-1}(\mathbb{R}^N))}) \\
&+ \sum_{j=1}^{N-1} (\|\tilde{h}_j\|_{L_p((0,t), W_q^2(\mathbb{R}^N))} + \|\partial_t \tilde{h}_j\|_{L_p((0,t), L_q(\mathbb{R}^N))}) + \|\tilde{d}\|_{L_p((0,t), W_q^2(\mathbb{R}^N))},
\end{aligned}$$

by (7.1) and (7.2) we have

$$\tilde{\mathbb{F}}_{p,q}(t) \leq C \mathbb{F}_{p,q}(t) \quad \text{for any } t \in (0, T). \quad (7.5)$$

By (1.11)

$$\tilde{f}_{\text{div}}|_{t=0} = 0, \quad \tilde{\mathbf{f}}_{\text{div}}|_{t=0} = 0, \quad \tilde{g}_i|_{t=0} = 0, \quad \tilde{h}_j|_{t=0} = 0 \quad (7.6)$$

for $i = 1, \dots, N-1$ and $N+1$, and $j = 1, \dots, N-1$. Given function f defined on $(0, T)$, the extension operator $E_t f$ is defined by

$$[E_t f](\cdot, s) = \begin{cases} f_0(\cdot, s) & -\infty < s < t, \\ f_0(\cdot, 2t - s) & t < s < \infty, \end{cases} \quad (7.7)$$

where f_0 is the zero extension of f to $(-\infty, 0)$, that is $f_0(\cdot, s) = f(\cdot, s)$ for $0 < s < t$ and $f_0(\cdot, s) = 0$ for $-\infty < s < 0$. Obviously, $[E_t f](\cdot, s) = 0$ for $s \notin [0, 2t]$. Moreover, if $f|_{t=0} = 0$, then

$$\partial_s [E_t f](\cdot, s) = \begin{cases} 0 & -\infty < s < 0, \\ \partial_s f(\cdot, s) & 0 < s < t, \\ -(\partial_s f)(\cdot, 2t - s) & t < s < 2t, \\ 0 & 2t < s < \infty. \end{cases} \quad (7.8)$$

Instead of (7.4), we consider the equations:

$$\begin{aligned}
\rho_{*+} \partial_t \mathbf{w}_+ - \text{Div } \mathbf{S}_{*+}(\mathbf{w}_+) &= \tilde{\mathbf{f}}_+ && \text{in } \mathbb{R}_+^N \times \mathbb{R} \\
\rho_{*-} \partial_t \mathbf{w}_- - \text{Div } \mathbf{S}_{*-}(\mathbf{w}_-) + \nabla p_- &= \tilde{\mathbf{f}}_-, \quad \text{div } \mathbf{w}_- = \tilde{f}_{\text{div}} = \text{div } \tilde{\mathbf{f}}_{\text{div}} && \text{in } \mathbb{R}_-^N \times \mathbb{R} \\
\mu_{*-} D_{iN}(\mathbf{w}_-)|_- - \mu_{*+} D_{iN}(\mathbf{w}_+)|_+ &= \tilde{g}_i \quad (i = 1, \dots, N-1) && \text{in } \mathbb{R}_0^N \times \mathbb{R},
\end{aligned}$$

$$\begin{aligned}
(\mu_{*-}D_{NN}(\mathbf{w}_-) - p_-)|_- &= \frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}} \Delta' J && \text{in } \mathbb{R}_0^N \times \mathbb{R}, \\
(\mu_{*+}D_{NN}(\mathbf{w}_+) + (\nu_{*+} - \mu_{*+})\text{div } \mathbf{w}_+ \mathbf{I})|_+ &= \frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}} \Delta' J + \tilde{g}_{N+1} && \text{in } \mathbb{R}_0^N \times \mathbb{R}, \\
w_{-i}|_- - w_{+i}|_+ &= \tilde{h}_i \quad (i = 1, \dots, N-1) && \text{in } \mathbb{R}_0^N \times \mathbb{R}, \\
\partial_t J - \left(\frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}} w_{-N}|_- - \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}} w_{+N}|_+ \right) &= \tilde{d} && \text{in } \mathbb{R}_0^N \times \mathbb{R}.
\end{aligned} \tag{7.9}$$

Let \mathcal{L} and \mathcal{L}^{-1} be the Laplace transform with respect to t and its inverse transform defined by

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(\cdot, t) dt, \quad \mathcal{L}^{-1}[g(\cdot, \lambda)](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda t} g(\gamma + i\tau) d\tau$$

with $\lambda = \gamma + i\tau \in \mathbb{C}$. If we denote the Fourier transform with respect to t and its inverse transform by \mathcal{F}_t and \mathcal{F}_τ^{-1} , then we have $\mathcal{L}[f](\cdot, \lambda) = \mathcal{F}_t[e^{-\gamma t} f](\tau)$ and $\mathcal{L}^{-1}[g(\cdot, \lambda)](t) = e^{\gamma t} \mathcal{F}_\tau^{-1}[g(\cdot, \gamma + i\tau)](t)$. Let $\mathcal{A}_\pm(\lambda)$, $\mathcal{P}_-(\lambda)$ and $\mathcal{H}(\lambda)$ be \mathcal{R} -bounded solution operators of problem (2.1) given in Theorem 2.2. If we apply the Laplace transform with respect to t to (7.9), then we have the generalized resolvent equation (2.1) with

$$\begin{aligned}
\mathbf{f}_\pm &= \mathcal{L}[E_t \tilde{\mathbf{f}}_\pm], \quad \tilde{f}_- = \mathcal{L}[E_t \tilde{f}_{\text{div}}], \quad \tilde{\mathbf{f}}_- = \mathcal{L}[E_t \tilde{\mathbf{f}}_{\text{div}}], \\
g_m &= \mathcal{L}[E_t \tilde{g}_m], \quad g_N = 0, \quad g_{N+1} = \mathcal{L}[E_t \tilde{g}_{N+1}], \quad h_m = \mathcal{L}[E_t \tilde{h}_m].
\end{aligned}$$

Thus, setting

$$\mathbf{w}_\pm = \mathcal{L}^{-1}[\mathcal{A}_\pm(\lambda) \mathbf{G}_\lambda], \quad p_- = \mathcal{L}^{-1}[\mathcal{P}_-(\lambda) \mathbf{G}_\lambda], \quad J = \mathcal{L}^{-1}[\mathcal{H}(\lambda) \mathbf{G}_\lambda]$$

with

$$\begin{aligned}
\mathbf{G}_\lambda &= (\mathcal{L}[E_t \tilde{\mathbf{f}}_+], \mathcal{L}[E_t \tilde{\mathbf{f}}_-], \lambda^{1/2} \mathcal{L}[E_t \tilde{f}_{\text{div}}], \nabla \mathcal{L}[E_t \tilde{f}_{\text{div}}], \lambda \mathcal{L}[E_t \tilde{\mathbf{f}}_{\text{div}}], \\
&\quad \lambda^{1/2} \mathcal{L}[E_t \tilde{\mathbf{g}}], \nabla \mathcal{L}[E_t \tilde{\mathbf{g}}], \lambda \mathcal{L}[E_t \tilde{\mathbf{h}}], \lambda^{1/2} \nabla \mathcal{L}[E_t \tilde{\mathbf{h}}], \nabla^2 \mathcal{L}[E_t \tilde{\mathbf{h}}]),
\end{aligned}$$

where $\tilde{\mathbf{g}} = (\tilde{g}_1, \dots, \tilde{g}_{N-1}, 0, \tilde{g}_N)$ and $\tilde{\mathbf{h}} = (\tilde{h}_1, \dots, \tilde{h}_{N-1})$, we see that \mathbf{w}_\pm , p_- and J satisfy the equations (7.9) and the estimate

$$\begin{aligned}
\mathcal{E}_\gamma &\leq C \{ \|e^{-\gamma t} E_t \tilde{\mathbf{f}}_+\|_{L_p(\mathbb{R}, L_q(\mathbb{R}_+^N))} + \|e^{-\gamma t} (E_t \tilde{\mathbf{f}}_-, \Lambda_\gamma^{1/2} E_t \tilde{f}_{\text{div}}, \nabla E_t \tilde{f}_{\text{div}}, \partial_t E_t \tilde{\mathbf{f}}_{\text{div}})\|_{L_p(\mathbb{R}, L_q(\mathbb{R}_+^N))} \\
&\quad + \|e^{-\gamma t} (\Lambda_\gamma^{1/2} E_t \tilde{\mathbf{g}}, \nabla E_t \tilde{\mathbf{g}}, \partial_t E_t \tilde{\mathbf{h}}, \Lambda_\gamma^{1/2} \nabla E_t \tilde{\mathbf{h}}, \nabla^2 E_t \tilde{\mathbf{h}})\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))} \}.
\end{aligned} \tag{7.10}$$

for any $\gamma \geq \gamma_0$ with some constants C and γ_0 with the help of Weis's operator valued Fourier multiplier theorem [17], where we have set $\Lambda_\gamma^{1/2} = \mathcal{L}^{-1}[\lambda^{1/2} \mathcal{L}[f]]$ and

$$\begin{aligned}
\mathcal{E}_\gamma &= \sum_{\ell=\pm} \|e^{-\gamma t} (\partial_t \mathbf{w}_\ell, \Lambda_\gamma^{1/2} \nabla \mathbf{w}_\ell, \nabla^2 \mathbf{w}_\ell)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}_\ell^N))} + \|e^{-\gamma t} \nabla p_-\|_{L_p(\mathbb{R}, L_q(\mathbb{R}_+^N))} \\
&\quad + \|e^{-\gamma t} (\partial_t J, \nabla J)\|_{L_p(\mathbb{R}, W_q^2(\mathbb{R}^N))},
\end{aligned}$$

and we have used the fact that

$$(\partial_t, \Lambda_\gamma^{1/2} \nabla, \nabla^2) \mathbf{w}_\pm = \mathcal{L}^{-1}[G_\lambda^1 \mathcal{A}_\pm(\lambda) \mathbf{G}_\lambda], \quad \nabla p_- = \mathcal{L}^{-1}[\nabla \mathcal{P}_-(\lambda) \mathbf{G}_\lambda], \quad (\partial_t, \nabla) J = \mathcal{L}^{-1}[G_\lambda^2 \mathcal{H}(\lambda) \mathbf{G}_\lambda].$$

As was seen in Shibata and Shimizu [13], we know that

$$\gamma \|e^{-\gamma t} f\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C \|e^{-\gamma t} \partial_t f\|_{L_p(\mathbb{R}, L_q(\Omega))}, \tag{7.11}$$

$$\begin{aligned}
\|\Lambda_\gamma^{1/2} f\|_{L_p(\mathbb{R}, L_q(\Omega))} &\leq C \{ \|\partial_t f\|_{L_p(\mathbb{R}, W_q^{-1}(\Omega))} + \|f\|_{L_p(\mathbb{R}, W_q^1(\Omega))}, \\
\|e^{-\gamma t} \partial_t f\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\gamma t} f\|_{L_p(\mathbb{R}, W_q^2(\Omega))} &\leq C \|e^{-\gamma t} (\partial_t f, \Lambda_\gamma^{1/2} \nabla f, \nabla^2 f)\|_{L_p(\mathbb{R}, L_q(\Omega))}
\end{aligned} \tag{7.12}$$

for any $\gamma \geq \gamma_0$ with some constant $C > 0$, where $\Omega = \mathbb{R}_\pm^N$ or $= \mathbb{R}^N$. Thus, combining (7.10) and (7.12), we have

$$\mathcal{E}_\gamma + \sum_{\ell=\pm} \gamma \|e^{-\gamma t} \mathbf{w}_\pm\|_{L_p(\mathbb{R}, L_q(\mathbb{R}_\ell^N))} + \gamma \|e^{-\gamma t} J\|_{L_p(\mathbb{R}, W_q^2(\mathbb{R}^N))} \leq C \mathcal{M}_\gamma \tag{7.13}$$

for any $\gamma \geq \gamma_0$ with some constant $C > 0$ with

$$\begin{aligned} \mathcal{M}_\gamma = & \sum_{\ell=\pm} \|e^{-\gamma t} E_t \tilde{\mathbf{f}}_\ell\|_{L_p(\mathbb{R}, L_q(\mathbb{R}_\ell^N))} + \|e^{-\gamma t} \partial_t E_t \tilde{f}_{\text{div}}\|_{L_p(\mathbb{R}, W_q^{-1}(\mathbb{R}_\pm^N))} + \|e^{-\gamma t} E_t \tilde{f}_{\text{div}}\|_{L_p(\mathbb{R}, W_q^1(\mathbb{R}_\pm^N))} \\ & + \|e^{-\gamma t} \partial_t E_t \tilde{\mathbf{f}}_{\text{div}}\|_{L_p(\mathbb{R}, L_q(\mathbb{R}_\pm^N))} + \|e^{-\gamma t} \partial_t E_t \tilde{\mathbf{g}}\|_{L_p(\mathbb{R}, W_q^{-1}(\mathbb{R}^N))} + \|e^{-\gamma t} E_t \tilde{\mathbf{g}}\|_{L_p(\mathbb{R}^N, W_q^{-1}(\mathbb{R}^N))} \\ & + \|e^{-\gamma t} \partial_t E_t \tilde{\mathbf{h}}\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))} + \|e^{-\gamma t} E_t \tilde{\mathbf{h}}\|_{L_p(\mathbb{R}, W_q^2(\mathbb{R}^N))} \end{aligned}$$

for any $\gamma \geq \gamma_0$ with some constant $C > 0$. Especially, it follows from (7.13) and the fact that $E_t f$ vanishes for $t < 0$ that \mathbf{v}_\pm and J also vanish for $t < 0$. In fact, we observe that

$$\begin{aligned} & \gamma(\|\mathbf{w}_+\|_{L_p((-\infty, 0), L_q(\mathbb{R}_+^N))} + \|\mathbf{w}_-\|_{L_p((-\infty, 0), L_q(\mathbb{R}_-^N))} + \|J\|_{L_p((-\infty, 0), W_q^2(\mathbb{R}^N))}) \\ & \leq \gamma(\|e^{-\gamma t} \mathbf{v}_+\|_{L_p(\mathbb{R}, L_q(\mathbb{R}_+^N))} + \|e^{-\gamma t} \mathbf{v}_-\|_{L_p(\mathbb{R}, L_q(\mathbb{R}_-^N))} + \|e^{-\gamma t} J\|_{L_p(\mathbb{R}, W_q^2(\mathbb{R}^N))}) \leq C\mathcal{M}_\gamma \leq CM_{\gamma_0} \end{aligned}$$

for any $\gamma \geq \gamma_0$, so that letting $\gamma \rightarrow \infty$ we have $\|\mathbf{w}_\pm\|_{L_p((-\infty, 0), L_q(\mathbb{R}_\pm^N))} = 0$ and $\|J\|_{L_p((-\infty, 0), W_q^2(\mathbb{R}^N))} = 0$, which implies that \mathbf{w}_\pm and J vanish for $t < 0$. By the equation, we also have that ∇p_- vanishes for $t < 0$. Summing up, we have seen that $\mathbf{w}_\pm \in L_p(0, \infty), W_q^2(\mathbb{R}_\pm^N)) \cap W_p^1((0, \infty), L_q(\mathbb{R}_\pm^N))$, $p_- \in L_p((0, \infty), \dot{W}_q^1(\mathbb{R}_-^N))$ and $J \in L_p((0, \infty), W_q^3(\mathbb{R}^N)) \cap W_p^1((0, \infty), W_q^2(\mathbb{R}^N))$, \mathbf{w}_\pm , p_- and J satisfy (7.4) with $T = t$, because $E_t f = f$ for $t \in (0, t)$. Moreover, by (7.7), (7.8), and (7.13), we have

$$\mathcal{I}_{p,q}(\mathbf{w}_\pm, p_-, J)(t) \leq C e^{\gamma t} \tilde{\mathbb{F}}_{p,q}(t) \leq C e^{\gamma t} \mathbb{F}_{p,q}(t). \quad (7.14)$$

Since the uniqueness follows from the existence of solutions to the dual problem, the equation (7.4) admits unique solutions for $(0, t)$. Especially, we can construct unique solutions of problem (7.4) for $(0, T)$ and these solutions are also solutions of problem (7.4) with $T = t$ for any $0 < t < T$, so that by (7.14) we have completed the proof of Theorem 1.1.

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